A SINGULAR INFINITE DIMENSIONAL HAMILTON-JACOBI-BELLMAN EQUATION ARISING FROM A STORAGE PROBLEM

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Abstract. In the first part of this paper, we derive an infinite dimensional partial differential equation which describes an economic equilibrium in a model of storage which includes an infinite number of non-atomic agents. This equation has the form of a mean field game master equation. The second part of the paper is devoted to the mathematical study of the Hamilton-Jacobi-Bellman equation from which the previous equation derives. This last equation is both singular and set on a Hilbert space and thus raises new mathematical difficulties.

Contents

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1. Introduction

In this paper, we study a particular infinite dimensional Hamilton-Jacobi-Bellman (HJB in short) equation which arises in the modeling of an economic equilibrium problem. This problem, of a new type, arises from he interaction of a large number of facilities of storage between each other. The description of this model is the subject of the first part of this paper while the second one is concerned with a mathematical analysis of the HJB equation yielded by this modeling part.

The model which produces the equation of interest is concerned with the storage of a good in an infinite number of sites. At each site, an equilibrium takes place between supply, demand, storage and carriers who can bring the good from one site to the other.

Writing the equilibrium equations for the transfer of goods leads to an equation of the form of a Mean Field Game (MFG for short) master equation. We refer to [16, 7, 2, 3] for more details on MFG master equations and to [15, 8, 9] for more on MFG. Let us insist on the fact that the MFG master equation arising from our modeling is not of the exact nature of most of the master equations studied in the literature because i) no precise game between the players is written, ii) the state variable is not the repartition of agents in the state space but rather the repartition of the product (or the good) in the state space. The fact that master equations can describe economic equilibrium outside of the usual MFG setting has already been remarked in [5, 1] and we believe it is a general feature of equilibrium models.

Since we are considering an infinite number of sites, the associated master equation is set on a Hilbert space. Quite remarkably, this equation derived from a HJB equation, which is thus naturally associated to the problem of a social planner. We focus our mathematical study on this singular HJB equation on a Hilbert space.

The study of HJB equations on Hilbert space dates back to [11, 12, 13, 19]. Quite recently it has attracted quite a lot of attention, see for instance [14, 6, 10]. In this paper, we provide a study of our equation relying on finite dimensional approximations of the limit problem.

2. A toy model for storage in a large number of separated units

2.1. **Description of the model.** There are $N \geq 2$ sites, one good and several populations of agents. On each site, there are local consumers and suppliers, local storage, and arbitrageurs. There are also carriers who transfer the good from site n to site $n+1$, and vice versa ; moreover, $N+1$ is identified with 1.

These agents interact through local markets, one market at each site. We denote by $p_{n,t}$ the price of the good on the site n at time t, and by $k_{n,t}$ the level of storage of the good on the site n at time t. In our model, we assume that $k_{n,t}$ can take any positive or negative value. In the real world, storage is bounded : the low bound is often 0 while the upper bound is due to physical or technical capacities. Moreover, in many cases, for operational reasons, there is a high targeted level of storage. In these cases, our level of storage of $k_{n,t}$ would represent the difference between the actual level of storage and this high targeted level. Introducing lower and upper bounds for the storage, our model would become both more realistic, more interesting and much more difficult from a mathematical viewpoint as was highlighted in [1]. However the focus of the present paper is different, we want to show the effect that a multitude of sites of storage can have on the price.

Apart from shocks which are described below, the consumption flows are supposed to be given by demand functions $D_n(p_{n,t})$ and the supply flows are supposed to be given by supply functions $S_n(p_{n,t})$. Additionally we assume that there are shocks on supply and demand so that the net supply, during a time interval of length dt, is given by

$$(S_n(p_{n,t}) - D_n(p_{n,t}))dt + \sigma dW_{n,t}, \tag{2.1}$$

where $(W_n)_{n\geq 1}$ is a collection of $i.i.d.$ Brownian motions on a standard probability space.

2.2. **The equilibrium equations.** We denote by $f_{n,t}$ the, algebraic, flow of transfer from n to $n+1$ at time t. We assume that the global cost of transfer is given by $\frac{c}{2}f_{n,t}^2$. Transfers are assumed to be fast enough, so that at equilibrium, the marginal cost of transfer is equal to the difference of prices in sites n and $n+1$

$$cf_{n,t} = p_{n+1,t} - p_{n,t}. \tag{2.2}$$

Arbitrageurs own the stored goods. The global cost of storage is given by a function $g_n(k_{n,t})$. Arbitrageurs are assumed to be risk neutral and to discount their future revenues at the rate $r \geq 0$, hence, at equilibrium the following holds

$$p_{n,t} = \mathbb{E}[p_{n,t+dt}]e^{-rt} - g_n'(k_{n,t}), \tag{2.3}$$

where the expectations is taken with respect to the collection of independent Brownian motions. Indeed, if the price $p_{n,t}$ is lower than the right hand side, then the arbitrageurs are incentivized to buy at time t, pay the cost to store the good, and then sell it at time $t+dt$. Note that the previous relation indeed holds because there is no constraint on the storage. If there were state constraints on the storage, then the situation would be closer to the one studied in [1], and only an inequality would hold when $k_{n,t}$ reaches a constraint.

In this economy, the state of the world is the level of storage at each sites $K = (k_n)_{1\leq n\leq N}$. Hence, we look for an equilibrium described by prices $(p_n)_{1\leq n\leq N}$ of the form $p_n = p_n(K)$. Thanks to (2.3), such functions p_n have to satisfy at equilibrium

$$
(2.4)\qquad \begin{aligned} 0 = -rp_n(K) + \frac{\sigma^2}{2}\sum_{i=1}^{N}\partial_{k_ik_i}p_n(K) - g_n'(k_n) + \\ + \sum_{i=1}^{N}\partial_{k_i}p_n(K)\left[S_n(p_n) - D_n(p_n) + \frac{p_{n+1}+p_{n-1}-2p_n}{c}\right]. \end{aligned}
$$

Let us remark that equation (2.4) has the typical form of a MFG master equation [4, 2]. However, contrary to the usual MFG master equations, the variable K does not describe here the repartition of players in the state space but rather the repartition of objects in the state space (here the level of storage in each site). An interpretation of this fact is that, in this model, the repartition of objects is more important than the repartition of players, such was already in the case in [5, 1] for instance.

This fact is not casual but rather general : MFG master equation can appear in equilibrium models with a state variable which is not the repartition of players in the state space.

Finally, since there is no friction in our market equilibrium model, not surprisingly, invisible hand principle applies and the MFG master equation (2.4) can be derived from a HJB equation which is the HJB equation of a benevolent planner of the invisible hand. We detail this fact in the next section.

2.3. **The limit equations.** Here we derive, formally, the limit of the previous equation as $N \to \infty$. Since p depends on the level of storage in every sites $K \in \mathbb{R}^d$, we expect that taking $N \to \infty$, p depends on a function $k : [0,1] \to \mathbb{R}$. Moreover, since p also depends on the site, it is also a function of a real variable x. We assume that p is at least defined on $k \in L^2([0,1])$. Hence, assuming some integrability of $p(k)$ for any $k \in L^2$, the price function p can be described as $p : L^2([0,1]) \to L^2([0,1])$.

To simplify the following, we assume that the costs of storage do not depend on n and that they are given by an affine function g, thus it no longer appears in the term D_i. We assume also that $F_n(p) = p$ for all n. The proper scaling for the passage to the limit $N \to \infty$ is $c = N^{-2}$. If we assume that the sites are located uniformly on a unit circle, this corresponds to a quadratic cost of transportation. In this regime, the limit of (2.4) becomes

$$
(2.5)\qquad 0 = -rp(k) + \frac{\sigma^2}{2}\Delta_k p(k) + \langle p(k) - \Delta_x p(k), \nabla\rangle p(k) + g(k), \text{ in } L^2([0,1]).
$$

In the previous equation, the operator Δ_k is the Laplacian operator in the space of function taking values in $L^2([0,1])$, i.e., it is the trace of the Hessian; $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2([0,1])$, Δ_x is the usual Laplacian operator from $H^2([0,1])$

into $L^2([0,1])$ and ∇ is the usual gradient operator on the Hilbert space $L^2([0,1])$.

Using these notations, assume that $\phi : L^2([0,1]) \to \mathbb{R}$ is a smooth solution of the equation

$$r\phi - \frac{\sigma^2}{2}\Delta_k\phi - \frac{1}{2}\langle (Id - \Delta_x)\nabla\phi, \nabla\phi\rangle = G(k) \text{ in } L^2([0,1]), \tag{2.6}$$

where $G : L^2([0,1]) \to \mathbb{R}$ is such that $\nabla G(k) = g(k)$. Then, $\nabla\phi$ is a solution of (2.5). Hence, solving (2.5) can be done by means of the HJB equation (2.6).

Obviously, the study of (2.6) is easier than the one of (2.5). The interest of the latter, is that it offers a wider range of applications. Indeed, some models can be expressed by equation of the form of (2.5) without the possibility of integrating the equation into an HJB equation such as (2.6). This is for instance the case for general storage costs g. However, for the sake of simplicity, we limit ourselves to the study of a time dependent counterpart of (2.6) in this paper.

To conclude this modeling part, we make the analogy with the MFG terminology. The equation (2.5) is the master equation associated to this problem, its solution helps to describe the underlying equilibria. In the potential case, the equation (2.6) is the HJB equation of the social planner. We focus on this point of view here.

3. Notation and mathematical formulation of the problem

Let $(H, \langle\cdot,\cdot\rangle)$ be a separable real Hilbert space and $A \in \mathcal{L}(H)$ a symmetric, positive and invertible operator such that A^{-1} is compact. We denote by $(\lambda_n)_{n\geq 0}$ the increasing sequence of eigenvalues of A (possibly with repetition according to the multiplicity) and by $(e_n)_{n\geq 0}$ an orthonormal basis of H formed of corresponding eigenvectors. For the rest of the paper, we shall adopt the convention

$$\forall x \in H, x = \sum_{i=0}^{\infty} x_i e_i, \text{ where } (x_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}}. \tag{3.1}$$

We consider the following PDE whose solution is denoted by $\phi : (0,\infty)\times H \to \mathbb{R}$

$$\partial_t\phi - \Delta\phi + \frac{1}{2}\langle A\nabla\phi, \nabla\phi\rangle = 0, \text{ in } (0,\infty)\times H, \tag{3.2}$$

$$\phi|_{t=0} = \phi_0 \text{ in } H. \tag{3.3}$$

In the previous equations, ϕ_0 is considered as a data from the model and assumptions on it shall be made later on, ∇ is the gradient operator and Δ is the Laplcian, i.e. it is the trace of the Hessian operator, when it is defined. The aim of this paper is to provide a suitable notion of solution for the singular PDE (3.2)-(3.3).

Remark 3.1. *The case of equation* (3.2) *with a right hand side term could have also been treated. Although, because it is merely an extension of the study we present here, we leave this trivial extension to the interested reader.*

3.1. **Preliminary results.**

3.2. **The quadratic case.** Let us consider first the instructive case in which ϕ_0 is given by

$$\phi_0(x) = \frac{1}{2}\sum_{i=1}^{\infty} \mu_i^0 x_i^2, \tag{3.4}$$

for some bounded real sequence $(\mu_n^0)_{n\geq 0} \in \ell^\infty$. We then seek a solution of (3.2)-(3.3) of the form

$$\phi(t,x) = c(t) + \frac{1}{2}\sum_{i=1}^{\infty} \mu_i(t) x_i^2, \text{ for } t \geq 0, x \in H. \tag{3.5}$$

When plugging this function in (3.2)-(3.3), one obtain that ϕ is indeed a solution if and only if for all $i \geq 0$

$$\begin{cases} \frac{1}{2}\dot{\mu}_i(t) + \frac{1}{2}\lambda_i \mu_i(t)^2 = 0, \\ \mu_i(0) = \mu_i^0, \end{cases} \tag{3.6}$$

and

$$\dot{c}(t) = \sum_{i=1}^{\infty} \mu_i(t). \tag{3.7}$$

From this we deduce that if for some $i \geq 0$, $\mu_i^0 < 0$, then there is no solution to (3.2)-(3.3) as there is explosion in finite time. On the other hand, if $\mu_i^0 \geq 0$ for all $i \geq 0$, then the systems (3.6) lead to

$$\forall i \geq 0, t \geq 0, \mu_i(t) = \frac{\mu_i^0}{1 + \lambda_i \mu_i^0 t}. \tag{3.8}$$

This naturally yields a formula for c which is

$$\forall t \geq 0, c(t) = \sum_{i=0}^{\infty} \lambda_i^{-1} \log(1 + \lambda_i \mu_i^0 t). \tag{3.9}$$

The previous is well defined only if $\sum_i \lambda_i^{-1}\log(1+\lambda_i) < +\infty$. The previous computations can be summarized in

Proposition 3.1. *Assume that* $\sum_i \lambda_i^{-1}\log(1+\lambda_i) < +\infty$. *For any positive bounded sequence* μ^0, *the function* ϕ *defined by* (3.5)-(3.9) *is a classical solution of* (3.2), *which satisfies the appropriate initial condition.*

The question of uniqueness of such a solution is treated later on. Let us insist upon the fact that, in this simple setting, the sequence $(\lambda_i)_{i\geq 0}$ has to satisfy

$$\sum_i \lambda_i^{-1} \log(1+\lambda_i) < +\infty, \tag{3.10}$$

and ϕ_0 has to be assumed to be convex.

3.3. **The deterministic case.** Let us consider the deterministic problem

$$\begin{aligned} \partial_t \psi + \frac{1}{2}\langle A\nabla\psi, \nabla\psi\rangle &= 0, \text{ in } (0,\infty)\times H, \\ \psi|_{t=0} &= \phi_0 \text{ in H.} \end{aligned} \tag{3.11}$$

Formally the solution of this equation is given by the Lax-Oleinik formula

$$\psi(t,x) = \inf_{y\in H}\left\{\phi_0(y) + \frac{\langle A^{-1}(x-y), x-y\rangle}{2t}\right\}. \tag{3.12}$$

Of course it is not clear that this formula makes sense without assumptions on A and ϕ_0. Nonetheless, we believe that the deterministic case is insightful for the study of (3.2). For instance, it helps us to understand the behavior of the solution of (3.2) near $t=0$. Indeed the function ψ defined by (3.12) is not continuous in t at $t=0$ without additional assumptions on the initial condition. The following result gives a necessary condition for which this is the case.

Proposition 3.2. *Assume that ϕ_0 is convex, continuous and that $\phi_0(x)\to\infty$ when $|x|\to\infty$. Then for any $x\in H$, $\psi(t,x)\to\phi_0(x)$ as $t\to 0$ where ψ is defined by* (3.12).

Proof. Let us observe first that $\psi(t,x)\leq\phi_0(x)$ for all t,x (simply choose $y=x$ in (3.12)). Fix $x\in H$ and consider a sequence $(t_n)_{n\geq 0}$ of non negative elements which converges toward 0. Consider for all $n\geq 0$ an element $y_n\in H$ such that

$$\phi_0(y_n) + \frac{\langle A^{-1}(x-y_n), x-y_n\rangle}{2t_n} \leq \psi(t_n,x) + \frac{1}{n+1} \leq \phi_0(x) + \frac{1}{n+1}. \tag{3.13}$$

From the growth assumption on ϕ_0, we deduce that, up to a subsequence, $(y_n)_{n\geq 0}$ has a weak limit $y^*\in H$. From (3.13), since ϕ_0 is clearly bounded from below, we deduce that $y^* = x$. Since ϕ_0 is lower semi-continuous for the weak topology, we deduce finally the desired result. □

Another situation in which the deterministic case is helpful is the one in which we have estimates on the Laplacian of a solution ϕ of (3.2). In this case, we can use the solution given by (3.12) (if it is well defined) to obtain some continuity estimates near $t=0$ on ϕ as the next result explains.

Proposition 3.3. *Let us consider a smooth solution ϕ of* (3.2) *such that $\phi_{|t=0}$ is continuous, convex and goes to ∞ as $|x|\to\infty$ and that $\Delta\phi$ is uniformly bounded,* *i.e.* $\|\Delta\phi\|_\infty \leq c$. *Moreover consider the function* ψ *defined by* (3.12) *with* $\phi_0 = \phi|_{t=0}$. *Assume moreover that*

$$|\phi(t,x) - \psi(t,x)| \underset{|x|\to\infty}{=} o(\psi(t,x)). \tag{3.14}$$

Then the following holds for $t \geq 0, x \in H$

$$\psi(t,x) - ct \leq \phi(t,x) \leq \psi(t,x) + ct. \tag{3.15}$$

Let us comment on (3.14). This assumption controls the distance between ψ and ϕ. For instance, if the difference of these two functions is bounded, then (3.14) holds. Indeed recall that ψ is a coercive function of x for any t. Moreover, let us insist on the fact that the bounds obtained in the previous Proposition do not depend quantitatively on this assumption, but only on the fact that it holds. This shall be helpful later on in the paper.

Remark 3.2. *Let us remark that if* ψ *is convex, then one can replace the conclusion of the Proposition by, for all* $t \geq 0, x \in H$ *:*

$$\psi(t,x) \leq \phi(t,x) \leq \psi(t,x) + ct. \tag{3.16}$$

Remark 3.3. *If instead of the uniform bound on the Laplacian, one has indeed* $\|\Delta\phi(t,\cdot)\|_\infty \leq c(t)$ *for some integrable function* $c(\cdot)$. *Then one can replace the conclusion of the Proposition with*

$$\psi(t,x) - \omega(t) \leq \phi(t,x) \leq \psi(t,x) + \omega(t), \tag{3.17}$$

where $\omega(t) = \int_0^t c(s)ds$.

Proof. We only prove the second inequality, the first one can be obtained in a similar fashion. The following is by now somehow standard in the study of Hamilton-Jacobi equations in infinite dimension.
Without loss of generality we can assume that $\phi|_{t=0} \geq 0$. Indeed, the equation is invariant by the addition of a constant and $\phi|_{t=0}$ is bounded from below. We want to prove that for any $\lambda \in (0,1)$, $t \geq 0, x \in H$,

$$\lambda\phi(t,x) \leq \psi(t,x) + ct. \tag{3.18}$$

Let us assume that (3.18) does not hold. Hence there exists $(t_*, x_*) \in (0,\infty) \times H$ such that $\lambda\phi(t_*,x_*) < \psi(t_*,x_*) + ct_*$. Thus there exists $\delta > 0$, such that $w : (t,x) \to \psi(t,x) + ct - \lambda\phi(t,x) + \delta t$ satisfies $w|_{t=0} \geq 0$ and $w(t_*,x_*) < 0$. From this, we deduce that t_0 defined by

$$t_0 = \inf\{t > 0, \exists x \in H, w(t,x) < 0\} \tag{3.19}$$

is smaller than t_*. From the coercivity of $\psi(t_0)$ and (3.14), we deduce that $w(t_0)$ is also coercive. Hence, there exists $R > 0$ such that

$$\inf_{|x|\leq R} w(t_0,x) = 0. \tag{3.20}$$

From Stegall's Lemma [17, 18], which is always true in Hilbert spaces, we deduce that for any $\epsilon > 0$, there exists $\xi \in H$, $|A\xi| \leq \epsilon$ and $x_0 \in H$ such that x_0 is a strict minimum of $x \to w(t_0, x) + \langle \xi, x\rangle$ on $\{x \in H, |x| \leq R\}$. By construction of t_0 and x_0, the following relations hold

$$
\begin{cases} \partial_t w(t_0, x_0) \leq 0, \\ \nabla\psi(t_0, x_0) + \xi = \lambda\nabla\phi(t_0, x_0). \end{cases} \tag{3.21}
$$

In the previous, the second line comes from the computation of the gradient of w. Recall that

$$
\partial_t\psi(t_0, x_0) + \frac{1}{2}\langle A\nabla\psi(t_0, x_0), \nabla\psi(t_0, x_0)\rangle = 0. \tag{3.22}
$$

Using (3.21), we deduce that

$$
\lambda\partial_t\phi(t_0, x_0) - \delta - c + \frac{1}{2}\langle A(\lambda\nabla\phi(t_0, x_0) - \xi), \lambda\nabla\phi(t_0, x_0) - \xi\rangle \geq 0. \tag{3.23}
$$

From the assumption we made on ϕ, we can use the PDE it is a solution of to obtain

$$
\lambda\Delta\phi(t_0, x_0) - \delta - c + (\lambda^2 - \lambda)\frac{1}{2}\langle A\nabla\phi(t_0, x_0), \nabla\phi(t_0, x_0)\rangle + \lambda\langle A\nabla\phi(t_0, x_0), \xi\rangle + \frac{1}{2}\langle A\xi, \xi\rangle \geq 0. \tag{3.24}
$$

Recall now that A is a positive operator, hence $\langle A\nabla\phi(t_0, x_0), \nabla\phi(t_0, x_0)\rangle \geq 0$. Thus we obtain that, taking ϵ small enough (ϵ was used to measure the size of ξ), we obtain a contradiction in the previous inequality. Hence for any $\lambda \in (0, 1)$, (3.18) holds. The required result is then proved by passing to the limit $\lambda \to 1$ in (3.18).

□

3.4. **Counterexamples to the time continuity of the solution of the deterministic problem.** We present two examples in which ϕ_0 does not satisfy the assumptions of the previous result to justify in some sense that some assumptions have to be made.

The first one is an example in which ϕ_0 does not satisfy the required growth assumptions at infinity.

Example 3.1. *Assume that the eigenvalues of A are given by $\lambda_n = (n+1)^\alpha, n \geq 0$ and consider $x^* \in H$ given by $x^*_n = (n+1)^{-\beta}, n \geq 0$. Let us choose $\phi_0(x) = \langle x, x^*\rangle^2$. We take α and β such that $\alpha > 2\beta > 1$. The following holds for any $x \in H$*

$$
\inf_{y\in H}\left\{\phi_0(y) + \frac{1}{2t}\langle A^{-1}(x - y), x - y\rangle\right\} \underset{t\to 0}{\longrightarrow} 0. \tag{3.25}
$$

Indeed fix $x \in H$ and consider $z \in H$ defined by

$$\begin{cases} z_k = x_k, k \leq N-1 \\ z_N = -(x_N^*)^{-1} \sum_{i=0}^{N-1} x_i^* z_i, \\ z_k = 0, k \geq N+1 \end{cases} \tag{3.26}$$

where N is to be fixed later on. We compute

$$\begin{aligned} \phi_0(z) + \frac{1}{2t}\langle A^{-1}(x-z), x-z\rangle &= \frac{1}{2t}\left(\lambda_N^{-1}(z_N - x_N)^2 + \sum_{i \geq N+1} \lambda_i^{-1} x_i^2\right), \\ &\leq \frac{1}{2t}\left(\lambda_N^{-1}(z_N - x_N)^2 + \sum_{i \geq N+1} \lambda_i^{-1} x_i^2\right), \\ &\leq \frac{C}{2t}\left((N+1)^{-\alpha}(N+1)^{2\beta} + (N+1)^{1-\alpha}\right). \end{aligned} \tag{3.27}$$

where C is a constant depending on α, β and x. Then, since $\alpha > 2\beta > 1$, choosing N such that $(N+1)^{2\beta-\alpha}t^{-1}$ is as small as we want, we conclude that (3.25) *indeed holds and thus that the initial condition is not satisfied in such a case.*

The second example is a case in which ϕ_0 is not convex. This non-convexity highlights the non weak lower semi continuity of ϕ_0 which is crucial and which we also comment in the following example.

Example 3.2. *Consider ϕ_0 defined by*

$$\phi_0(x) = \begin{cases} |x|^2 \text{ if } |x| \geq 1, \\ 2 - |x|^2 \text{ else.} \end{cases} \tag{3.28}$$

For any $t > 0$, one has $\psi(t, 0) = 1$. Indeed, one necessary has $\psi(t, x) \geq \inf_{x \in H} \phi_0(x) = 1$. Then considering the sequence $y_n = e_n$ as a minimizing sequence in (3.12)*, it follows that $\psi(t, 0) \leq 1$, hence the fact that $\psi(t, 0) = 1$. This is an obvious contradiction to the fact that ψ converges toward ϕ_0 as $t \to 0$.*

More generally, it is the weak lower semi continuity of ϕ_0 which is important to obtain the time continuity in $t = 0$ of the deterministic solution. More precisely, let us assume that there is a sequence $(x_n)_{n \geq 0}$ weakly converging toward $x \in H$ such that both $\liminf \phi_0(x_n) < \phi_0(x)$ *and $(A^{-1}x_n)_{n \geq 0}$ converge (strongly) toward $A^{-1}x$. Then for any $t \geq 0$, $\psi(t, x) = \liminf \phi_0(x_n) < \phi_0(x)$.*

3.5. **A priori estimates.** In this section we present some key a priori estimates to establish existence and uniqueness of solutions of (3.2). Since we are not going to use those estimates directly, we prove them for a very specific class of functions. Nevertheless, those estimates are insightful on why the equation (3.2) is well posed.

For a smooth function $\phi : H \to \mathbb{R}$ and $i, j \geq 0$, we shall denote by ϕ_i the derivative of ϕ in the direction of e_i and by ϕ_{ij} the derivative of ϕ_i in the direction of e_j.

We define the space $\mathcal{B}$ of functions $[0,\infty) \times H \to \mathbb{R}$ such that

- $\phi \in \mathcal{C}^4$.
- For all $i,j \geq 0, \Delta\phi_i, \Delta\phi_{ij}$ are well defined.
- For all $i \geq 0$, $t \geq 0$, $\sum_{k\geq N} \partial_{kk}\phi_{ii}(t,x) \to 0$ as $N \to \infty$, uniformly in $x \in H$.
- $\nabla\phi, \nabla\phi_i, \nabla\phi_{ij} \in Dom(A)$.
- $\forall t \geq 0, \|\phi_{ii}(t,\cdot)\|_\infty < \infty$.

We acknowledge that this space $\mathcal{B}$ can seem a bit arbitrary at first. It is a suitable set in which the following a priori estimates are not too difficult to prove and in which any smooth function on the finite dimensional space H_N can be lifted. We shall come back more clearly on this second point later on in the paper.

As we already mentioned, the regularity of solutions of (3.2) can be established with a key a priori estimate on the second order derivatives of the solution that we now present.

Proposition 3.4. *If $\phi \in \mathcal{B}$ is a solution of* (3.2)*, then for all $t > 0$, $i \geq 0$, $x \in H$,*

$$\phi_{ii}(t,x) \leq \frac{\phi^0_{ii}}{1+\lambda_i\phi^0_{ii}t}, \tag{3.29}$$

where $\phi^0_{ii} := \|\phi_{ii|t=0}\|_\infty$. If $\phi_{|t=0}$ is convex, then $\phi(t)$ is convex for all time. If $\phi_{|t=0}$ is bounded from below by a constant, then ϕ is bounded from below by the same constant.

Remark 3.4. *If $\phi^0_{ii} = +\infty$, we understand the right hand side of* (3.29) *as simply $(\lambda_i t)^{-1}$.*

Proof. First let us remark that constants are solution of (3.2), thus if $\phi_{|t=0}$ is bounded from below by a constant, then ϕ is bounded from below by the same constant, using a comparison principle type result. This can be proved in exactly the same fashion as Proposition 3.3 and thus we do reproduce the argument here for the sake of clarity.

For all $i,j \geq 0$, ϕ_{ij} is a solution of

$$\partial_t\phi_{ij} - \Delta\phi_{ij} + \langle A\nabla\phi, \nabla\phi_{ij}\rangle + \sum_k \lambda_k\phi_{ki}\phi_{kj} = 0 \text{ in } (0,\infty)\times H. \tag{3.30}$$

Choosing $i = j$ in the previous equation yields

$$\partial_t\phi_{ii} - \Delta\phi_{ii} + \langle A\nabla\phi, \nabla\phi_{ii}\rangle + \lambda_i\phi^2_{ii} \leq 0 \text{ in } (0,\infty)\times H. \tag{3.31}$$

Assume now that (3.29) does not hold and thus that there exists, t_*, x_* such that

$$\phi_{ii}(t_*,x_*) > \frac{\phi^0_{ii}}{1+\lambda_i\phi^0_{ii}t_*}. \tag{3.32}$$

Let us define $u(t,x) = \phi_{ii}(t,x) - \frac{\phi_{ii}^0}{1+\lambda_i\phi_{ii}^0 t} - \delta t - \epsilon|x|^2$. By construction of (t_*, x_*), there exists $\delta, \epsilon > 0$ such that $u(t_*, x_*) > 0$. We consider such a pair (δ, ϵ). Let us now define, for some $N \geq 1$, t_0 with

$$t_0 = \inf\{t > 0, \exists x \in H_N, u(t,(x,0)) > 0\}.^1 \tag{3.33}$$

It is clear that t_0 depends on N, however let us recall that it is bounded in N since u is continuous in H. Since ϕ_{ii} is bounded (indeed $\phi \in \mathcal{B}$), we know that there exists $R > 0$ such that

$$\sup\{u(t_0,(x,0)), x \in H_N, |x| \leq R\} = 0. \tag{3.34}$$

Let us insist on the fact that $R > 0$ can be chosen independently of N here. Hence, using once again Stegall's Lemma [17, 18], we know that for any $\epsilon' > 0$ (independent of N), there exists $\xi \in H_N$, $|A\xi| \leq \epsilon'$ such that $x \to u(t,x) + \langle \xi, x\rangle$ has a strict maximum at some point $x_0 \in H_N$. Hence at we deduce the following

$$\begin{cases} u(t_0,x_0) = O(\epsilon'), \\ \partial_t\phi_{ii}(t_0,(x_0,0)) \geq \delta - \lambda_i\left(\frac{\phi_{ii}^0}{1+\lambda_i\phi_{ii}^0 t_0}\right)^2, \\ \nabla_N\phi_{ii}(t_0,(x_0,0)) = 2\epsilon x_0 - \xi, \\ \sum_{k=1}^N \partial_{kk}\phi_{ii} \leq 2\epsilon N. \end{cases} \tag{3.35}$$

Evaluating (3.31) at $(t_0,(x_0,0))$, we obtain
(3.36)

$$\delta - \lambda_i\left(\frac{\phi_{ii}^0}{1+\lambda_i\phi_{ii}^0}\right)^2 - 2\epsilon N - \sum_{k\geq N+1}\partial_{kk}\phi_{ii} + \langle A\nabla_N\phi, 2\epsilon x_0 - \xi\rangle + \lambda_i\phi_{ii}^2(t_0,(x_0,0)) \leq 0.$$

Using the first line of (3.31), we deduce

$$\delta - 2\epsilon N - \sum_{k\geq N+1}\partial_{kk}\phi_{ii} + O(\epsilon|x_0|) + O(\epsilon') \leq 0. \tag{3.37}$$

Let us now remark that, as usual with such viscosity solutions like estimates, $\epsilon|x_0| \to 0$ as $\epsilon \to 0$. Moreover, using the uniform summability of $\Delta\phi_{ii}$, we take N large enough, then we take $\epsilon \to 0$ and then we take $\epsilon' \to 0$ to arrive at a contradiction in the previous equation. Thus the claim follows.

□

We then deduce the following.

Corollary 3.1. *If $\phi \in \mathcal{B}$ is a smooth solution of (3.2) with convex initial data, then for any $t > 0$, $i, j \geq 0$*

$$|\phi_{ij}(t)| \leq \sqrt{\frac{\phi_{ii}^0}{1+\lambda_i\phi_{ii}^0 t}}\sqrt{\frac{\phi_{jj}^0}{1+\lambda_j\phi_{jj}^0 t}} \tag{3.38}$$

---

[1] Let us recall that $(x,0)$ denotes the element of H whose N first components are equal to x and the rest are set to 0.

This corollary simply follows from the non-negativity of the Hessian matrix of ϕ as well as form the bounds on ϕ_{ii}.

We now establish the extension of a classical estimate on the gradient of a convex function by its second order derivatives in finite dimension.

Proposition 3.5. *Assume $\phi \in \mathcal{B}$ is a smooth solution of* (3.2) *with convex initial data. Assume also that ϕ_0 is bounded from below by some constant. Then there exists a positive function $M : (0,\infty) \times H \to \mathbb{R}$ depending only on ϕ_0 and A such that*

$$|\nabla\phi(t,x)| \leq M(t,x)\sqrt{\phi(x) - \inf_H \phi_0}. \tag{3.39}$$

This type of inequalities is more or less standard . We include a proof for the sake of completeness.

Proof. Let us consider $x \in H$ and consider the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(\theta) = \phi(x + \theta\nabla\phi(x)). \tag{3.40}$$

Let us observe that ψ is a smooth function and that

$$\psi'(\theta) = \langle \nabla\phi(x + \theta\nabla\phi(x)), \nabla\phi(x)\rangle, \tag{3.41}$$

$$\psi''(\theta) = \langle \nabla\phi(x) D^2\phi(x + \theta\nabla\phi(x)), \nabla\phi(x)\rangle. \tag{3.42}$$

For any θ, the following holds

$$\psi(\theta) = \psi(0) + \psi'(0)\theta + \frac{\theta(\theta - z)}{2}\psi''(z), \tag{3.43}$$

for some $z \in [0,\theta]$. Hence

$$0 \leq (\psi(0) - \inf_H \psi) + \psi'(0)\theta + \frac{\theta^2}{2}\|\psi''\|_\infty. \tag{3.44}$$

This second order expression in θ does not change sign, thus

$$(\psi'(0))^2 \leq 2\|\psi''\|_\infty(\psi(0) - \inf_H \psi). \tag{3.45}$$

The required result then easily follows from this inequality.

□

3.6. **A formal change of variable.** We present a change of variable which simplifies the PDE (3.2). Formally if ϕ is a solution of (3.2) then defining v by

$$v(t, Bx) = \phi(t,x), \tag{3.46}$$

where $B = A^{-1/2}$. We observe that formally v satisfies

$$\partial_t v + \frac{1}{2}|\nabla v|^2 - Tr(B^2D^2v) = 0, \text{ in } (0,T) \times H \tag{3.47}$$

4. Existence and uniqueness of solutions

4.1. **Existence.** For any $N \geq 0$, let us consider the Hilbert space $H_N = Span(\{e_1, ..., e_N\})$. By construction H_N is stable under A. The N dimensional problem associated to (3.2) is

$$\partial_t \phi - \Delta_N \phi + \frac{1}{2}\langle A\nabla\phi, \nabla\phi\rangle = 0 \text{ in } (0, \infty) \times H_N, \tag{4.1}$$

where we used the notation Δ_N to insist on the fact that we are here in H_N. The unknown is $\phi : (0, \infty) \times H_N \to \mathbb{R}$. The idea we are going to follow in this section, is that, with an appropriate choice of initial conditions, the sequence of solutions of (4.1) converges toward a solution of (3.2), in a sense to be made precise later on.

We start with recalling the following classical result for problem (4.1).

Theorem 1. *Assume $\phi_0^N \in \mathcal{C}(H_N)$ is bounded from below, then there exists a unique viscosity solution ϕ_N of* (4.1) *with initial condition ϕ_0^N. Moreover ϕ_N is locally Lipschitz continuous and $\mathcal{C}^{1,1}$ in space.*

We now use in some sense the a priori estimates we presented earlier, or rather their proof as we are going to use them at the level of H_N. Before doing so, let us indicate the choice of the initial condition we use for (4.1). Several choices are suitable, we could for instance set $\phi_0^N(x) = \phi_0(x')$ for $x \in H_N$ where $x' \in H$ is such that $x_i = x'_i$ for $i \leq N$ and $x'_i = 0$ else. However, in all that follows, we set for $x \in H_N$

$$\phi_0^N(x) = \inf\{\phi_0((x, y)) | y \in H_N^{\perp}\}, \tag{4.2}$$

where (x, y) is the vector whose first $N+1$ coordinates are given by x and the other coordinates by y. We define the solution of the deterministic problem in the finite dimension case ψ^N which is given for $x \in H_N$ by

$$\psi^N(t, x) = \inf_{x' \in H_N} \left\{ \phi_0^N(x') + \frac{\langle A^{-1}(x - x'), x - x'\rangle}{2t} \right\}. \tag{4.3}$$

Let us remark that the following holds.

Proposition 4.1. *If ϕ_0 is continuous, convex and satisfies $\phi_0(x) \to \infty$ when $|x| \to \infty$, then ϕ_0^N and ψ^N converges toward respectively ϕ_0 and ψ*

For a fixed $N \geq 1$, we can establish the

Theorem 2. *For all $N \geq 0$, let ϕ^N be the solution of* (4.1) *with initial condition ϕ_0^N given by* (4.2)*. Assume that $\phi_0 \in \mathcal{C}^{1,1}$ is convex and satisfies $\phi_0(x) \to \infty$ as $|x| \to \infty$. Then for all $N \geq 0$, ϕ^N satisfies*

- *for all $0 \leq i, j \leq N, x \in H_N, t \geq 0$, $|(\phi^N)_{ij}(t, x)| \leq \frac{1}{\sqrt{\lambda_i \lambda_j} t}$.*
- $|\nabla \phi^N(t, x)| \leq M(t, x)\sqrt{\phi^N(x) - \inf_H \phi_0^N}$.
- $\psi^N \leq \phi^N \leq \psi^N + w(t)$.

- *If we define $v(t,Bx)=\phi^N(t,x)$ then v is a solution of*

$$\partial_t v+\frac{1}{2}|\nabla v|^2-\sum_{i=0}^{N}\lambda_i v_{ii}=0,\ \text{in } (0,T)\times H_N \tag{4.4}$$

Proof. We start by proving that ϕ_0^N is continuous, convex and satisfies $\phi_0^N(x)\to\infty$ when $|x|\to\infty$. The only point which is not immediate is the convexity and we detail it here. Let us take $\theta\in(0,1), x,x'\in H_N$. Since ϕ_0 is continuous and convex, it is lower semi continuous for the weak topology of H. Since it satisfies a growth condition, we deduce that there exists $y,y'\in H$ such that $\phi_0^N(x)=\phi_0((x,y))$ and $\phi_0^N(x')=\phi_0((x',y'))$. We now compute

$$\begin{aligned}\theta\phi_0^N(x)+(1-\theta)\phi_0^N(x')&=\theta\phi_0((x,y))+(1-\theta)\phi_0^N((x',y'))\\ &\geq\phi_0((\theta x+(1-\theta)x',\theta y+(1-\theta)y'))\\ &\geq\phi_0^N(\theta x+(1-\theta)x').\end{aligned}\tag{4.5}$$

From these properties of ϕ_0^N, we can establish that all the computations of the previous section are still valid and that ϕ^N satisfies the estimates of the Theorem, except for two points : ϕ^N is not necessary in $\mathcal{B}$, mainly because it is not $\mathcal{C}^4$ in space, and we have to ensure that the hypotheses of Proposition 3.3 hold.

The first point can be treated easily by approximation of the initial condition with $\mathcal{C}^4$ functions that we do not present here as they are standard. Moreover, extending the function ϕ^N to H by setting $\hat{\phi}^N(t,(x,x'))=\phi^N(t,x)$ allows to apply Propositions 3.3 and 3.4.

Concerning the second poitn, let us recall that since $\phi_0^N\in\mathcal{C}^{1,1}$, $\|\Delta\phi^N(t)\|_\infty\in L^1_{loc}((0,\infty),\mathbb{R})$ uniformly in N. Moreover, classical comparison principles hold and we can bound $\|\psi^N-\phi^N\|_\infty$ using the bound on $\|\Delta\phi^N(t)\|_\infty\in L^1_{loc}((0,\infty),\mathbb{R})$. Hence we can apply Proposition 3.3 and we know that $\psi^N\leq\phi^N\leq\psi^N+w(t)$ for some modulus of continuity w which is independent of N, thanks to Remark 3.3. ☐

The main result of this section follows.

Theorem 3. *Assume that ϕ_0 is convex $\mathcal{C}^{1,1}$ function which satisfies $\phi_0(x)\to\infty$ when $|x|\to\infty$. Then there exists a solution $\phi:[0,\infty)\times H\to\mathbb{R}$ of (3.2) in the following sense*

- *ϕ is continuous as an element of $\mathcal{C}([0,\infty),\mathcal{C}(H))$ and satisfies $\phi(0)=\phi_0$.*
- *The function v defined by (3.46) is a viscosity solution of (3.47)*
- *ϕ satisfies the a priori estimates of the previous section.*

Proof. Thanks to Theorem 2, we consider for any N the solution ϕ_N of (4.1) with initial condition (4.2). Define for any $N\geq 1$, $\hat{\phi}_N(t,(x,x'))=\phi_N(t,x)$ for $(x,x')\in H$ and $x\in H_N$. For any $t>0$, $(\hat{\phi}_N(t,\cdot))_{N\geq 0}$ is a sequence of bounded and uniformly in N Lipschitz functions. Indeed, this sequence satisfies uniformly the a priori estimates established in Section 3.5 as this was shown in Theorem 2. Hence, extracting a subsequence if necessary, it has a limit ϕ toward which it uniformly converges on all compact sets of H. The fact that ϕ satisfies the requirements of the Theorem is standard and we do not detail it here.

□

The previous result can be extended to more general initial conditions.

Corollary 4.1. *Assume that ϕ_0 can be approximated uniformly by $\mathcal{C}^{1,1}$ functions and that it is convex and satisfies $\phi_0(x) \to \infty$ when $|x| \to \infty$. Then the conclusions of Theorem 3 hold.*

Proof. Let us consider a uniform approximation $\tilde{\phi}_{0,\epsilon}$ of ϕ_0 (at distance at most ϵ). Denote by $\tilde{\phi}_{\epsilon,N}$ and ϕ_N the solution of (4.1) with respective initial conditions $\tilde{\phi}^N_{0,\epsilon}$ and ϕ^N_0. Let us note that ϕ_N satisfies all the conclusions of Theorem 2 except for the comparison with the solution of the deterministic problem.

Arguing as in the proof of Theorem 3, $(\tilde{\phi}_{\epsilon,N})_{N\geq 1}$ converges toward a a function $\tilde{\phi}_\epsilon$ which satisfies the requirements of Theorem 3. The only argument missing to apply directly the same argument to the sequence $(\phi_N)_{N\geq 1}$ is the bound. However, for $N \geq 1$, we can use a comparison principle in H_N to obtain that

$$\psi^N(t,x) \leq \phi_N(t,x) \leq \tilde{\phi}_{\epsilon,N}(t,x) + \epsilon. \tag{4.6}$$

Hence the previous argument implies that $(\phi_N)_{N\geq 1}$ has a limit ϕ which satisfies all the requirements of Theorem 3, except for the time continuity at the origin. To observe that this continuity indeed holds, it suffices to remark that we can compare the function $\tilde{\phi}_\epsilon$ to the solution of the associated deterministic problem ψ_ϵ. Hence the following holds

$$\psi(t,x) \leq \phi(t,x) \leq \psi_\epsilon(t,x) + w_\epsilon(t) + \epsilon. \tag{4.7}$$

From this we obtain that $\phi(t) \to \phi_0$ as $t \to 0$, since it holds for any $\epsilon > 0$ and since ψ_ϵ converges toward ψ as $\epsilon \to 0$. □

4.2. **Uniqueness.** The following uniqueness result can be established.

Theorem 4. *Under the assumptions of Corollary 4.1, there exists a unique solution ϕ of* (3.2) *in the sense that it satisfies all the conclusions of Theorem 3.*

Proof. Consider ϕ the solution provided by Theorem 3 (or its Corollary) and by $\tilde{\phi}$ another solution. Denoting by $\tilde{\phi}^N(t,x) := \tilde{\phi}(t,(x,0))$ for $t \geq 0, x \in H_N$, we immediately remark, thanks to the convexity of the solution, that for all $N \geq 1$, $\tilde{\phi}^N$ is a super solution of (4.1), hence

$$\phi^N \leq \tilde{\phi}^N \tag{4.8}$$

from classical comparison principles in finite dimension. Hence we obtain that $\phi \leq \tilde{\phi}$.

The opposite inequality is slightly more involved. Assume that there exists $t^* > 0, x \in H$ such that $\phi(t^*, x) + \delta < \tilde{\phi}(t^*, x)$ for some $\delta > 0$. For $N \geq 1$, let us introduce

$$\hat{\phi}^N(t,x) = \inf\{\tilde{\phi}(t,(x,x'))|x' \in H_N^{\perp}\} \tag{4.9}$$

and

$$\hat{v}^N(t,Bx) = \hat{\phi}^N(t,x) \quad ; \quad \tilde{v}(t,Bx) = \tilde{\phi}(t,x) \tag{4.10}$$

By construction, $\hat{v}^N$ satisfies

$$\partial_t \hat{v}^N + \frac{1}{2}|\nabla \hat{v}^N|^2 - \sum_{i=1}^{N} \lambda^{-i} \hat{v}^N_{ii} = \sum_{i \geq N+1} \lambda^{-i} \tilde{v}_{ii} \text{ in } (0,\infty) \times H_N. \tag{4.11}$$

Because, $\tilde{\phi}$ satisfies the a priori estimates, we know that $|\tilde{v}_{ii}| \leq t^{-1}$. Hence, for any $\kappa > 0$, there exists $N \geq 1$, such that, uniformly,

$$\left| \sum_{i \geq N+1} \lambda^{-i} \tilde{v}_{ii} \right| \leq \frac{\kappa}{t}. \tag{4.12}$$

Let us now define

$$t_N := \inf \left\{ t > 0, \exists x \in H_N, \phi^N(t,x) + \frac{\delta}{2} < \hat{\phi}^N(t,x) \right\}. \tag{4.13}$$

Since we assumed that t^* exists, for N sufficiently large, t_N is well defined. We want to show that $t_N \to 0$ as $N \to \infty$. If this is not the case, then, extracting a subsequence if necessary, for all $N \geq 1$, $t_N \geq \alpha$ for some $\alpha > 0$. By taking N larger than the one corresponding to $\kappa = \delta\alpha/(2(t^* - \alpha))$ in (4.12), we arrive at a contradiction. Hence, $t_N \to 0$ as $N \to \infty$.

In order to conclude, it suffices to remark that this is in contradiction with the continuity at $t = 0$ of both ϕ and $\tilde{\phi}$. Hence t^* does not exist, and thus $\phi \geq \tilde{\phi}$ which concludes this proof.

□

Acknowledgments

The three authors acknowledge a partial support from the Chair FDD from Institut Louis Bachelier.

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