MEAN FIELD GAMES WITH INCOMPLETE INFORMATION

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Abstract. This paper is concerned with mean field games in which the players do not know the repartition of the other players. First a case in which the players do not gain information is studied. Results of existence and uniqueness are proved and discussed. Then, a case in which the players observe the payments is investigated. A master equation is derived and partial results of uniqueness are given for this more involved case.

Contents

Introduction

This paper is interested in Mean Field Games (MFG in short) in which the players do not have a complete information on the repartition of the other players in the state space. Namely they are mostly unable to observe each other and have

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only an a priori on the initial distribution of players. This type of MFG leads to new mathematical questions which are partially solved here.

MFG have attracted quite a lot of attention since the seminal work [12, 13]. They are differential games involving non-atomic agents. MFG arise in a wide variety of modeling context such as Economics [11, 1], financial engineering [6], epidemiology [10] or Telecommunications [4]. For a vast majority of the existing literature, it is always assumed that all the players have a complete information on the MFG, i.e. they can observe at any time the state and action of each player. In this paper, cases in which all the information is not available to the players are studied.

Several authors have studied problems in which the players do not know directly their individual state but only have some partial information on it, see for instance [14]. This setup is different from the one we study here. P.-L. Lions studied a MFG in which all the players are learning an unknown parameter of the model. The closest work to ours in terms of models is [8] in which the authors studied a MFG in which the players do not know the controls of the other players, only the effect they have (as a whole) on their objective function. This last work relies on the fact that their model is explicitly solvable.

The rest of the paper is organized as follows. A presentation of the MFG model and a quick discussion on the structure of information in MFG is first. The rest of the paper is divided in two parts which make the core of the paper. The first one is concerned with a case in which the players have an incomplete initial information and do not gain any information with time. The second one is devoted to the situation in which the players do not observe the state of the other players but have complete information on their payments.

1. The MFG model and the classical structure of information

1.1. **Presentation of the model.** We present here the framework of the underlying game between the players. The state of each players is a process valued on the d dimensional torus $\mathbb{T}^d$ which evolves according to

$$dX_t = \alpha_t dt + \sqrt{2\sigma} dW_t, \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is a d dimensional Brownian motion on a standard (fixed) filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. The game lasts a time $T > 0$ and the cost of a player who uses the control $(\alpha_s)_{s\geq 0}$ is given by

$$\int_0^T f(X_s, m_s) + L(X_s, \alpha_s) ds + U_0(X_T, m_T), \tag{1.2}$$

where $(m_t)_{t\geq 0}$ is the evolution of the measure describing the spatial distribution of players. Clearly the cost paid by the players is unknown to them at the initial time since the evolution of their state is stochastic. We naturally assume that the players are risk neutral and take into account the expected cost they are too face which is, if the evolution $(m_t)_{t\in[0,T]}$ is known,

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T f(X_s, m_s) + L(X_s, \alpha_s)ds + U_0(X_T, m_T)\right]. \tag{1.3}$$

We do not particularly insist on why we make such an assumption, which is wildly common in the literature on stochastic optimal control. Hence, given an anticipation $(m_t)_{t\in[0,T]}$, a player can compute its optimal response by solving the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} -\partial_t u(t,x) - \sigma\Delta u(t,x) + H(x, \nabla_x u(t,x)) = f(m_t)(x) \text{ in } (0,T)\times\mathbb{T}^d,\\ u(T,x) = U_0(m_T)(x) \text{ in } \mathbb{T}^d, \end{aligned} \tag{1.4}$$

where we have introduced the Hamiltonian $H(x,p) := \sup_\alpha\{-\alpha\cdot p - L(x,\alpha)\}$. The associated optimal control is given in feedback form by

$$\alpha_t = -D_pH(X_t, \nabla_x u(t, X_t)). \tag{1.5}$$

On the other hand, given that the players use a strategy of the form $\alpha_t = b(t, X_t)$ for some function b, their repartition in the state space evolves according to the Fokker-Planck equation

$$\partial_t m_t - \sigma\Delta m_t + \operatorname{div}(bm_t) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \tag{1.6}$$

which is, as usual, understood in the sense of distribution. Hence, given an initial repartition of players $m_0 \in \mathcal{P}(\mathbb{T}^d)$, a strategic equilibrium is reached if one can find a solution (u, m, b) of (1.4)-(1.6) together with $b(t,x) = -D_pH(x, \nabla_x u(t,x))$. This is summarized in the system

$$\begin{aligned} &-\partial_t u - \nu\Delta u + H(x, \nabla u) = f(m)(x),\\ &\partial_t m - \nu\Delta m - \operatorname{div}(D_pH(x,\nabla u)m) = 0,\\ &u(T,x) = G(m(T))(x), m(0) = m_0, \end{aligned} \tag{1.7}$$

where the dependence of the unknown (u, m) in (t, x) is omitted to lighten the notation.

1.2. **The structure of information.** In the previous system, if neither the particular form of the second order term or the fact that dependence in m and ∇u are decoupled are important, a fundamental observation lies in the initial distribution of players m_0. This observation is that the knowledge of m_0 is equivalent (in terms of induced equilibria) to the knowledge of the whole evolution of the repartition of players $(m_t)_{t\in[0,T]}$. This quite simple fact do not need any particular proof as it only suffices to remark that m_0 is the only datum in the previous system of equations. Of course, it is a consequence of the deterministic evolution of $(m_t)_{t\in[0,T]}$, given the strategies of the players. In other words, even if the players do not observe each other during the game, as long as they know m_0, the induced equilibria are the same as if they observe the whole trajectory $(m_t)_{t\in[0,T]}$. Because the structure (or set) of equilibria only depends on the initial distribution m_0, the following question seems natural : what happens to the structure of equilibria if the players do not know m_0 ?

To answer this question, some assumptions have to be made on the knowledge at each instant that the players have on the repartition of other players. However, let us state that if the process $(m_t)_{t\geq 0}$ is not known and the players have a prior $(\mu_t)_{t\geq 0}$ on it, a risk neutrality assumption shall be made on the players. By prior we mean that instead of anticipating an evolution $(m_t)_{t\geq 0}$ for the repartition of players, the players believe that at any time $t \geq 0$, the repartition of players is unknown and that this uncertainty is described by the measure $\mu_t \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d))$. In such a context, the new expected cost of the players is

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T \int_{\mathcal{P}(\mathbb{T}^d)} f(X_s, m) d\mu_s + L(X_s, \alpha_s) ds + \int_{\mathcal{P}(\mathbb{T}^d)} U_0(X_T, m) d\mu_T\right]. \tag{1.8}$$

1.3. **Assumptions and notation.** We now present the standing assumptions for the rest of the paper. Before that, let us recall some some properties of sets of probability measures.

Assume that (E, d) is a compact metric space. Then, the set $\mathcal{P}(E)$, of Borel probability measures on E, can be equipped with the distance $\mathbf{d}_1$ defined by

$$\mathbf{d}_1(\mu, \nu) := \sup_{\phi} \int_E \phi(x)(\mu - \nu)(dx), \tag{1.9}$$

where the supremum is taken over Lipschitz functions on (E, d) with a Lipschitz constant of at most 1. The set $(\mathcal{P}(E), \mathbf{d}_1)$ is compact. In all this paper, $\mathcal{P}(E)$ is always seen as equipped with $\mathbf{d}_1$. In particular, $\mathcal{P}(\mathcal{P}(\mathbb{T}^d))$ is a compact set.

For the rest of the paper, we assume the following

- The hamiltonian H is smooth(, convex) and globally Lipschitz continuous in p, uniformly in x.
- The function f (respectively U_0) is continuous from $\mathcal{P}(\mathbb{T}^d)$ to $\mathcal{C}(\mathbb{T}^d)$ (respectively to $\mathcal{C}^2(\mathbb{T}^d)$).

Let us also recall that given a duality product $\langle \cdot, \cdot \rangle$ between two sets E and E', a mapping $F : E' \to E$ is said to be

- monotone if for all $x, y \in E'$

$$\langle F(x) - F(y), x - y \rangle \geq 0. \tag{1.10}$$

- strictly monotone if for all $x, y \in E'$

$$\langle F(x) - F(y), x - y \rangle = 0 \Rightarrow F(x) = F(y) \tag{1.11}$$

2. The blind case

The following situation shall be called the blind case. In this situation the players all start with an a priori $\mu_0 \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d))$ on the initial repartition of players and they do not gain any information during the game. By this we mean that they only observe their individual state for the whole duration of the game. If the players have the anticipation $(\tilde{\mu}_t)_{t\geq 0}$ for their prior, from (1.8), the optimization problem they have to face is described by the HJB equation

$$\begin{aligned} -\partial_t u - \sigma\Delta u + H(x, \nabla u) &= \int_{\mathcal{P}(\mathbb{T}^d)} f(m) d\tilde{\mu}_t \text{ in } [0,T]\times\mathbb{T}^d; \\ u(T) &= \int_{\mathcal{P}(\mathbb{T}^d)} U_0(m) d\tilde{\mu}_T \text{ in } \mathbb{T}^d, \end{aligned} \tag{2.1}$$

from which they can compute an optimal response given by $\alpha_t = -D_p H(X_t, \nabla_x u(t, X_t))$. Let us insist on the fact that, in this case, the players do not observe the cost they are paying, only their state. Indeed, in this model, the players start with an evolution of an a priori on the repartition of players, and they stick to this a priori during the game.

2.1. **Evolution of the belief of the players.** It remains to describe the evolution of the a priori of the players. Under an anticipation of $b : [0,T]\times\mathbb{T}^d \to \mathbb{R}^d$ for the strategies of the players, if the initial repartition of players is m_0, recall that its evolution can be computed through the Fokker-Planck equation

$$\partial_t m - \sigma\Delta m + \operatorname{div}(bm) = 0 \text{ in } (0,\infty)\times\mathcal{P}(\mathbb{T}^d), \tag{2.2}$$

with initial condition

$$m|_{t=0} = m_0. \tag{2.3}$$

Hence the evolution of the belief of the players is the push forward of the initial belief by this Fokker-Planck equation. To be more precise, denote the solution of (2.2) at time t with initial condition m_0 by $K_t(m_0)$. This define a semi group of operators $(K_t)_{t\geq 0}$. Given an initial belief μ_0 (and anticipations b), the belief μ_t at time t is given by

$$\mu_t = (K_t)_{\#}\mu_0, \tag{2.4}$$

where $f_{\#}\mu$ denotes the image measure by the map f of the measure μ. To keep a partial differential equation (PDE) point of view on this problem, the evolution $(\mu_t)_{t\geq 0}$ characterized by (2.4) is formally the solution of the following continuity equation

$$\begin{aligned} &\partial_t \mu + \nabla_m \cdot \Big((\sigma\Delta m - \nabla_x\cdot(mb))\mu \Big) = 0 \text{ in } (0,T)\times\mathcal{P}(\mathbb{T}^d), \\ &\mu|_{t=0} = \mu_0. \end{aligned} \tag{2.5}$$

The operator ∇_m is thought as a divergence operator on $\mathcal{P}(\mathbb{T}^d)$. This is only formal of course. This continuity equation on $\mathcal{P}(\mathbb{T}^d)$ states that the weight that μ_t puts on any element of $\mathcal{P}(\mathbb{T}^d)$ is transported along the paths generated by the Fokker-Planck equation (2.2).

Moreover, the equation (2.5) can be understood in the weak sense as the following result explains.

Proposition 2.1. *Fix* μ_0 *and let* $(\mu_t)_{t\geq 0}$ *be defined by* (2.4) *(for a given smooth function* b*). For any smooth function* $\phi : [0,T]\times\mathcal{P}(\mathbb{T}^d)\to\mathbb{R}$ *such that* $\phi(T)=0$*, the following holds*

$$\begin{aligned}\int_0^T\int_{\mathcal{P}(\mathbb{T}^d)}\left(-\partial_t\phi-\int_{\mathbb{T}^d}(\sigma\Delta m(x)-div(m(x)b(t,x)))\frac{\delta\phi}{\delta m}(t,m,x)dx\right)d\mu_t dt \\ -\int_{\mathcal{P}(\mathbb{T}^d)}\phi(0)\mu_0=0.\end{aligned} \tag{2.6}$$

Moreover, it is the unique process to satisfy the previous variational relation.

Proof. It suffices to compute for any $t\geq 0$

$$\begin{aligned}\int_{\mathcal{P}(\mathbb{T}^d)}\phi(t)d\mu_t&=\int_{\mathcal{P}(\mathbb{T}^d)}\phi(t)d(K_t)_{\#}\mu_0\\&=\int_{\mathcal{P}(\mathbb{T}^d)}\phi(t,K_tm)\mu_0(dm)\end{aligned} \tag{2.7}$$

and to remark that for any $t\geq 0, m\in\mathcal{P}(\mathbb{T}^d)$

$$\frac{d}{dt}\phi(t,K_tm)=\partial_t\phi(t,K_tm)+\int_{\mathbb{T}^d}(\sigma\Delta m-\mathrm{div}(bm))\frac{\delta\phi}{\delta m}(t,K_tm,x)dx. \tag{2.8}$$

Because ϕ is smooth, in particular its derivative with respect to m is smooth in x, the previous integral is well defined. Integrating (2.8) between 0 and T and using (2.7) gives the first part of the result.

The second part is obtained by taking two such processes and by considering their difference ν which satisfies for any ϕ as in the statement

$$0=\int_0^T\int_{\mathcal{P}(\mathbb{T}^d)}\left(-\partial_t\phi-\int_{\mathbb{T}^d}(\sigma\Delta m(x)-\nabla_x\cdot(m(x)b(t,x)))\frac{\delta\phi}{\delta m}(t,m,x)dx\right)d\nu_t dt. \tag{2.9}$$

Take any smooth function $G:[0,T]\times\mathcal{P}(\mathbb{T}^d)\to\mathbb{R}$ and define

$$\phi(t,m):=\int_t^T G(s,K_{s-t}m)ds. \tag{2.10}$$

Now observe that by construction of ϕ, plugging it in (2.9) yields

$$\int_0^T\int_{\mathcal{P}(\mathbb{T}^d)}G(t,m)\nu_t(dm)dt=0. \tag{2.11}$$

Hence $\nu=0$ by density of smooth functions in $\mathcal{C}(\mathcal{P}(\mathbb{T}^d))$. □

Remark 2.1. *The question of existence of such a process has already been answered since it has been constructed above.*

Moreover, this evolution of the belief is continuous with respect to time, given that $b \in L^\infty$.

Proposition 2.2. *Assume that the drift b of* (2.2) *is bounded. Then $(\mu_t)_{t\in[0,T]}$ defined in* (2.4) *is uniformly $\frac{1}{2}$-Hölder continuous, with a constant depending only on $\|b\|_\infty$.*

Proof. By definition, for $s, t \in [0,T]$

$$\begin{aligned} \mathbf{d}_1(\mu_s,\mu_t) :&= \sup_{\|\phi\|_{Lip}\leq 1} \int_{\mathcal{P}(\mathbb{T}^d)} \phi d(\mu_s - \mu_t) \\ &= \int_{\mathcal{P}(\mathbb{T}^d)} \phi(K_s m) - \phi(K_t m) d\mu_0. \end{aligned} \tag{2.12}$$

Using both the classical $\frac{1}{2}$-Hölder continuity estimate on the Fokker-Planck equation and the Lipschitz continuity of ϕ we obtained the required result. □

Example 1. *To illustrate the previous evolution of the belief, consider the case in which μ_0 is a combination of Dirac masses. If it is given by $\mu_0 := n^{-1}\sum_{i=1}^n \delta_{m_i}$, then for any $t \geq 0$, μ_t is simply given by $\mu_t := n^{-1}\sum_{i=1}^n \delta_{K_t m_i}$.*

2.2. **Existence of Nash equilibria of the game.** In the same way as (1.7) characterizes Nash equilibria of the MFG when the initial of distribution of players is known, we can characterize Nash equilibria of the MFG with incomplete information with a system of PDE. Indeed given an initial belief μ_0 and a profile of strategy $b : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ for the players, one can compute the associated anticipation on the belief with (2.5). Hence, a best response is derived through the HJB equation (2.1) whose solution is u. One indeed gets a strategic equilibrium if $b(t,x) = -D_pH(x, \nabla_x u(t,x))$.

Thus Nash equilibria of the blind game with initial distribution μ_0 are characterized as solutions of

$$\begin{aligned} &-\partial_t u - \sigma\Delta u + H(x,\nabla u) = \int_{\mathcal{P}(\mathbb{T}^d)} f(m) d\mu_t \text{ in } [0,T]\times\mathbb{T}^d; \\ &\partial_t \mu + \nabla_m \cdot \Big((\sigma\Delta m + \nabla_x \cdot (m D_p H(\cdot, \nabla_x u))\mu \Big) = 0 \text{ in } (0,T)\times\mathcal{P}(\mathbb{T}^d), \\ &u(T) = \int_{\mathcal{P}(\mathbb{T}^d)} U_0(m) d\mu_T \text{ in } \mathbb{T}^d, \mu|_{t=0} = \mu_0. \end{aligned} \tag{2.13}$$

To lighten the notation, we introduce $\tilde{f} : \mu \in \mathcal{M}(\mathcal{P}(\mathbb{T}^d)) \to \int_{\mathcal{P}(\mathbb{T}^d)} f(x,m')\mu(dm') \in \mathcal{C}(\mathbb{T}^d)$ and analogously $\tilde{U}_0 : \mu \to \int_{\mathcal{P}(\mathbb{T}^d)} U_0(x,m')\mu(dm')$.

The following holds.

Theorem 2.1. *There exists a solution (u,μ) of* (2.13) *in the sense that u is a classical solution of the Hamilton-Jacobi-Bellman equation and μ is the unique solution of the continuity equation in the sense of Proposition 2.1.*

Proof. Let us consider the applications ψ_1, ψ_2 and ψ_3 defined by : $\psi_1 : \mathcal{C}([0,T], \mathcal{P}(\mathcal{P}(\mathbb{T}^d))) \to \mathcal{C}^{1,2}([0,T] \times \mathbb{T}^d)$ associates to $(\mu_t)_{t\geq 0}$ the solution of the HJB equation (2.1) ; $\psi_2 : \mathcal{C}^{1,2}([0,T] \times \mathbb{T}^d) \to \mathcal{C}([0,T], \mathcal{P}(\mathcal{P}(\mathbb{T}^d)))$ associates to a function u the solution of (2.5) with initial condition μ_0 and drift $b = -D_pH(\nabla_x u)$; $\psi_3 := \psi_2 \circ \psi_1$.

To prove that ψ_3 has a fixed point, using Schauder's fixed point Theorem, it is sufficient to establish that ψ_2 is a compact and continuous mapping as ψ_1 is clearly continuous.

The fact that ψ_2 is compact is a direct application of Proposition 2.2. The fact that it is continuous is a consequence of the continuity of the Fokker-Planck (2.2) equation with respect to the drift term b and on the definition of ψ_3 using (2.4). ☐

The previous result of existence of such equilibria is in no sense surprising and falls in a category of somehow classical result of existence of Nash equilibria in MFG. A more interesting feature is the effect the lack of knowledge has on uniqueness properties of the equilibria.

2.3. **Uniqueness of Nash equilibria.** Let us first compute the usual proof of uniqueness of MFG which dates back to the original paper of Lasry and Lions. Take two solutions (u_1, μ_1) and (u_2, μ_2) of the system (2.13). Take the difference of the left hand sides of the HJB equations, integrate against an arbitrary measure $m \in \mathcal{P}(\mathbb{T}^d)$ and then integrate once again against the difference $\mu_1 - \mu_2$, doing this, we obtain using the equations satisfied by μ_1 and μ_2

(2.14)

$$\int_0^T \int_{\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d} ((-\partial_t - \sigma\Delta)(u_1 - u_2) + H(x, \nabla u_1) - H(x, \nabla u_2)) m(dx)(\mu_1(t) - \mu_2(t))(dm)dt$$

$$= \int_0^T \int_{\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d} (\nabla_x(u_1 - u_2)D_pH(x, \nabla u_1) + H(x, \nabla u_1) - H(x, \nabla u_2))\, m(dx)d\mu_1(t)dt$$

$$+ \int_0^T \int_{\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d} (\nabla_x(u_2 - u_1)D_pH(x, \nabla u_2) + H(x, \nabla u_2) - H(x, \nabla u_1))\, m(dx)d\mu_2(t)dt$$

$$- \int_{\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d} \tilde{U}_0(\mu_1(T) - \mu_2(T))(x)m(dx)(\mu_1(T) - \mu_2(T))(dm).$$

Using the convexity of the Hamiltonian, and the HJB equations, we obtain

(2.15)

$$\int_{\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d} \tilde{U}_0(\mu_1(T) - \mu_2(T))(x)m(dx)(\mu_1(T) - \mu_2(T))(dm)$$

$$+ \int_0^T \int_{\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d} \tilde{f}(\mu_1(t) - \mu_2(t))(x)m(dx)(\mu_1(t) - \mu_2(t))(dm)dt \leq 0.$$

This somehow classical computation yields a uniqueness result which is analogous to the usual result of uniqueness for MFG Nash equilibria.

Theorem 2.2. *Assume that f and U_0 are such that*

$$\int_{\mathcal{P}(\mathbb{T}^d)}\int_{\mathbb{T}^d}\tilde{U}_0(x,\mu_1-\mu_2)m(dx)(\mu_1-\mu_2)(dm)\geq 0,\forall \mu_1,\mu_2\in\mathcal{P}(\mathcal{P}(\mathbb{T}^d)), \tag{2.16}$$

$$\int_{\mathcal{P}(\mathbb{T}^d)}\int_{\mathbb{T}^d}\tilde{f}(x,\mu_1-\mu_2)m(dx)(\mu_1-\mu_2)(dm)\leq 0\Rightarrow \tilde{f}(\mu_1)=\tilde{f}(\mu_2),\forall \mu_1,\mu_2\in\mathcal{P}(\mathcal{P}(\mathbb{T}^d)). \tag{2.17}$$

Then there is at most one solution (u,μ) of (2.13).

However, as we shall see, these monotone-like assumptions are way stronger than their classical counterparts.

Proposition 2.3.
- *Any function f which satisfy the requirements of the previous Theorem is a monotone operator for the L^2 scalar product.*
- *For any smooth function $\phi(t,x)$, if f is defined by*

$$f(t,x,m)=\phi(t,x)\int_{\mathbb{T}^d}\phi(t,y)m(dy), \tag{2.18}$$

  then it satisfies the assumption of the previous result.

Proof. The first claim follows immediately from choosing μ_1 and μ_2 as Dirac masses.

The second one follows from the computation

$$\begin{aligned}&\int_{\mathcal{P}(\mathbb{T}^d)}\int_{\mathbb{T}^d}\int_{\mathcal{P}(\mathbb{T}^d)}f(x,m')(\mu_1-\mu_2)(dm')m(dx)(\mu_1-\mu_2)(dm)\\&=\int_{\mathcal{P}(\mathbb{T}^d)}\int_{\mathbb{T}^d}\int_{\mathcal{P}(\mathbb{T}^d)}\phi(t,x)\int_{\mathbb{T}^d}\phi(t,y)m'(dy)(\mu_1-\mu_2)(dm')m(dx)(\mu_1-\mu_2)(dm)\\&=\left(\int_{\mathbb{T}^d}\phi(t,y)m'(dy)(\mu_1-\mu_2)(dm')\right)^2.\end{aligned} \tag{2.19}$$

□

If it clear that the previous example of existence of functions satisfying the requirements of Theorem 2.2 can be generalized (by adding terms independent of m for instance), it is also clear that this requirement is more restrictive than the monotonicity.

To highlight that, outside of these assumptions, the question of uniqueness is as difficult as for the usual MFG system, we present an example in which several equilibria are possible, in a two states model, in a non-dynamic setting, in a case in which a unique equilibria .

Example 2. *Consider a non-atomic game with two states A and B and a continuum of mass 1 of players. Initially the players do not know if they are distributed uniformly between A and B or if they are all in state A, each of the two possibility being anticipated with probability* $\frac{1}{2}$. *The players can change their state with a probability of their choice. The cost they face is the mass of players in the state in which they end up in (average with respect to the two possible scenarios). Denoting by* α *(respectively* β*) the proportion of players who change from state A to B (respectively from state B to A), a simple computation yields that there is a continuum of equilibria of the form* $(\alpha, \beta) \in \{(\frac{1}{3} + \epsilon, 3\epsilon), \epsilon \in (0, \frac{1}{3})\}$.

Remark 2.2. *Such a counterexample can be adapted in a continuous framework by setting* $\sigma = 0$, $f = 0$, $H(p) = |p|$, *T large enough so that the players can reach any point in the state space and by choosing the terminal cost* U_0 *appropriately.*

Remark 2.3. *Another quite simple, but rather important, information we can observe at the moment is that any monotone function f (in the* L^2 *sense) satisfies the requirements of the uniqueness result on the subset of* $\mathcal{P}(\mathcal{P}(\mathbb{T}^d))$ *defined by* $\mathcal{A} := \{\mu \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d)), \exists g, \mu \text{ a.e. in } m, f(m) = g\}$. *To put it more simply, if the indetermination on the initial distribution does not affect the payments, then the uniqueness argument on monotonicity is still valid.*

3. The case of observed payments

The last remark of the previous section suggests to be interested in the case in which, at any time, the absence of knowledge of the repartition of players does not translate into an absence of knowledge of the payments of the players. Indeed if at any time the players know the cost f, even if they do not know exactly the repartition of players m, the cost f "stays" monotone in some sense.

This section considers a case in which the players observe all the payments. By this we mean that at a time t, even if the players do not know exactly the actual repartition of players m_t, they know the cost $f(m_t)$ that it induces (on the whole state space). In this situation, the information of the costs is common to all the players. This structure is very reminiscent of a common noise in MFG. This section is far from being a complete study of such models and its aim is more to introduce this problem. Some partial results are given. The rest of this section is organized as follows. After a formal description of the model and some reminders on the disintegration of measures, the evolution of the belief with information on the payments is presented. We then derive the associated master equation and present a partial result of uniqueness.

It is worth mentioning that, in a situation in which the cost function is injective, such a model is of no interest as the players learn instantly the repartition of players. However we argue that in several models (especially macro-economic ones

[11]), the cost function is far from being one to one and, for instance, depends only on a few moments of the repartition of players.

3.1. **The model.** The framework is the following. Because the players are going to update their belief on the repartition of players using the information they have on the payments, this belief $(\mu_t)_{t\geq 0}$ can no longer be computed as a function of the strategies of the players and the initial belief μ_0. Even if $(\mu_t)_{t\geq 0}$ is by no mean random here, it is convenient to use the standard probabilistic framework to understand the object at interest here.

Even if the players do not know the initial distribution of players, there is an actual $m_0 \in \mathcal{P}(\mathbb{T}^d)$ which describes their initial repartition. Because the players have initially the belief μ_0, let us model m_0 as a $\mathcal{P}(\mathbb{T}^d)$ valued random variable whose law is μ_0. The evolution of the actual repartition of players is denoted $(m_t)_{t\geq 0}$. Because it is unknown to the players, it can also be modeled as a random variable whose law is $(\mu_t)_{t\geq 0}$. The players then observe at any time $t \geq 0$ the payments $f(m_t) : \mathbb{T}^{\mathrm{d}} \to \mathbb{R}$ and update their belief accordingly by conditioning it on their observation. The next section explains how the belief is updated.

Moreover, it seems clear that the information process $(f(m_t))_{t\geq 0}$ plays the role of a common noise and that a deterministic approach using a forward-backward system such as (2.13) can no longer be sufficient to model equilibria of the MFG and the master equation approach to characterize a value is presented later on. Refer to [5] for more details on MFG master equations. A similar approach to the one of [7] (used to deal with common noise) seems to be usable. However we were not able to adapt these arguments to non smooth conditionings.

3.2. **Reminders on disintegration.** Since players are going to update their belief according to the new information they gain, some facts on the disintegration, or conditioning, of measures are recalled. If one were to compute an expectation of some random variable, given an a priori information, the proper object to use would be the conditional expectation. However, a more precise object is the conditioning of the law of the random variable, given this information. The process of obtaining this conditioning is called in the literature the disintegration of a measure.

Given $\mu \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d))$, a function $\psi : \mathcal{P}(\mathbb{T}^d) \to E$, for $(E, \psi_{\#}\mu)$ a measured set, the disintegration of μ along ψ is a family $(\mu_y)_{y\in F}$ of probability measures on $\mathcal{P}(\mathbb{T}^d)$, where $F = \psi(\mathcal{P}(\mathbb{T}^d))$, such that

- For any measurable set $A \subset \mathcal{P}(\mathbb{T}^d)$, $\mu(A) = \int_{\mathcal{P}(\mathbb{T}^d)} \mu_{\psi(m)}(A)\mu(dm)$.
- For any $y \in F$, $\mu_y(\psi^{-1}(\{y\})) = 1$.

Formally, μ_y is the conditioning of μ on the fact that the information y has been received. Disintegrations could have been defined in more general settings, see section 452 of [9]. Their existence and uniqueness is an involved question. In this setting, the existence of a disintegration, and its uniqueness $\psi_{\#}\mu$ almost everywhere hold [15].

3.3. **Evolution of the belief.** This section describes the evolution of the belief of the players in the context of observed payments, given that the strategies of the players are given by a function $b : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$. First, recall that because at any time the players observe the payments, the process $(\mu_t)_{t\geq 0}$, which represents the common belief, has to be valued in

$$\mathcal{A} := \{\mu \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d)), \exists g, \mu \text{ a.e. in } m, f(m) = g\}. \tag{3.1}$$

Consider the total information the players have received up to the time $t \geq 0$, when the initial distribution of players is $m \in \mathcal{P}(\mathbb{T}^d)$. Denoting this information $\mathcal{F}(t,m)$, one finds

$$\mathcal{F}(t,m) := (f(K_s m))_{s\in[0,t]} \tag{3.2}$$

The belief of the players evolves as a combination of the two "rules"

(1) The process $(\mu_t)_{t\geq 0}$ is weighting elements of $\mathcal{P}(\mathbb{T}^d)$ which are transported along the **same** Fokker-Planc equation. Indeed, the strategies of the players, hence the drift b in (2.2), cannot depend on the different elements of $\mathcal{P}(\mathbb{T}^d)$ which are "weighted" by μ.
(2) At any time, the belief μ_t is disintegrated along the function $\mathcal{F}$ into $(\mu_\theta)_{\theta\in\mathcal{F}(\mathcal{P}(\mathbb{T}^d))}$ and the belief $\mu_{\theta'}$ which corresponds to the observed payments θ' becomes the new belief.

Obviously the previous is quite formal and a more precise definition is presented below.

Proposition 3.1. *Given $\bar{\mu} \in \mathcal{A}$, for $\bar{\mu}$ almost every m, there exists a process $(\mu_t^m)_{t\geq 0}$ which satisfies for any $t \geq 0$*

- $\mu_0^m = \bar{\mu}$.
- *For any $t \geq 0$, there exists ν_t such that $\mu_t^m = (K_t)_{\#}\nu_t$.*
- *For any $t \geq 0$, for μ_t^m almost every m', $f(m') = f(K_t m)$.*
- *For any measurable $A \subset \mathcal{P}(\mathbb{T}^d)$ such that for $\bar{\mu}$ almost every $m' \in A$, $f(K_t m') = f(K_t m)$, $\mu_t^m(K_t(A) \cup X) = \mu_t^m(X)$*

Moreover, for any $t \geq 0$, $\phi \in \mathcal{C}(\mathcal{P}(\mathbb{T}^d))$, $\int_{\mathcal{P}(\mathbb{T}^d)}\int_{\mathcal{P}(\mathbb{T}^d)} \phi(m')\mu_t^m(dm')\bar{\mu}(dm) = \int_{\mathcal{P}(\mathbb{T}^d)} \phi(m')(K_t)_{\#}\bar{\mu}(dm')$.

Remark 3.1. *This Proposition is almost the definition of the evolution of the belief in this model. The first point states the initial condition, the second one that this evolution follows the first rule above, the third one states that the belief is indeed consistent with the information and the fourth point states that the process $(\mu_t^{\mathcal{F}(m)})_{t\geq 0}$ is not too restrictive (not simply $(\delta_{K_t m})_{t\geq 0}$ for instance), even though no uniqueness result is stated here.*

Proof. For $t \geq 0$ consider the disintegration of μ along $\psi := \mathcal{F}(t, \cdot)$ and denote it by $(\tilde{\nu}_\theta)_{\theta \in \psi(\mathcal{P}(\mathbb{T}^d))}$. Defining $\nu_t := \tilde{\nu}_{\mathcal{F}(t,m)}$ and

$$\mu_t^{\mathcal{F}(m)} := (K_t)_\# \nu_t \tag{3.3}$$

proves the claim. Note that the disintegration is uniquely defined for $\psi_\# \bar{\mu}$ almost every θ [15]. $\square$

Even if this approach is only presented to formally derive the master equation associated to this problem, and thus not needed in details, the following example illustrates the previous result.

Example 3. *Consider $\bar{\mu}$ as combination of Dirac masses $\bar{\mu} := n^{-1} \sum_{i=1}^n \delta_{m_i}$. The information process is $\mathcal{F}(\cdot, m_1)$ defined for $f(m) = (\int_{\mathbb{T}^d} xm(dx) - K)_+$ for a certain level $K > 0$. Assume that for $1 \leq i \leq n, f(m_i) = 0$ and that the $(m_i)_{1 \leq i \leq n}$ are ordered such that the sequence of time $(t_i)_{2 \leq i \leq n}$ defined by*

$$t_i := \inf\{t > 0, f(K_t m_1) \neq f(K_t m_i)\} \tag{3.4}$$

satisfies $0 < t_n < t_{n-1} < ... < t_2 < t_1 = \infty$. Then for any $t \in (t_{k+1}, t_k]$ the belief given by Proposition 3.1 is $k^{-1} \sum_{i=1}^k \delta_{K_t m_i}$.

3.4. **Derivation of the master equation.** The associated master equation is derived formally in this section. We start by a development on functions on $[0, T] \times \mathcal{P}(\mathcal{P}(\mathbb{T}^d))$ of the form

$$V(t, \mu) := \mathbb{E}_\mu \left[\int_t^T G(\nu_s^{m,\mu}) ds \right], \tag{3.5}$$

for smooth G where $(\nu_t^{m,\mu})_{t \geq 0}$ is the process constructed in the proof of Proposition 3.1 when $\bar{\mu} \leftarrow \mu$. For any $dt > 0$,

$$V(t, \mu) = \mathbb{E}_\mu \left[\int_t^{t+dt} G(\nu_s^{m,\mu}) ds + V(t + dt, \nu_{t+dt}^{m,\mu}) \right]. \tag{3.6}$$

If the evolution of $(\nu_s^{m,\mu})_{s \geq t}$ has no reason to be smooth, because the information is precisely through f, for μ almost every m, $(f(\nu_s^{m,\mu}))_{s \geq t}$ is smooth. Hence if G is given as

$$G(\mu) := \int_{\mathcal{P}(\mathbb{T}^d)} \Psi(f(m)) \mu(dm) \tag{3.7}$$

for a smooth function Ψ, then

$$(dt)^{-1} \int_t^{t+dt} G(\nu_s^{m,\mu}) ds \underset{dt \to 0}{\longrightarrow} G(\mu). \tag{3.8}$$

This leads, formally, to the PDE

$$-\partial_t V - A[\mu, b, f][V] = G(\mu), \tag{3.9}$$

where $A[\mu, b, f][V]$ is the operator "defined" by

$$A[\mu, b, f][V] := \lim_{dt \to 0} \mathbb{E}_\mu \left[\frac{V(\nu_{dt}^{m,\mu}) - V(\mu)}{dt} \right], \tag{3.10}$$

where $(\nu_t^{m,\mu})_{t \geq 0}$ is the process given by Proposition 3.1, when the starting belief is μ and m the initial, unknown, repartition of players.

Clearly, because the evolution of $(\nu_t^{m,\mu})_{t \geq 0}$ is neither smooth nor even well defined, the domain of definition of A is not clear at all. We shall come back on this question later on.

Fixing the strategies of the "other" players through the function $b : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$, the value function of a player is

$$U(t, x, \mu) = \inf_\alpha \mathbb{E}_{\mathbb{P},\mu} \left[\int_t^T \int_{\mathcal{P}(\mathbb{T}^d)} f(X_s, m')\nu_s^{m,\mu}(dm') + L(X_s, \alpha_s) ds + \int_{\mathcal{P}(\mathbb{T}^d)} U_0(X_T, m')\nu_T^{m,\mu}(dm') \right]. \tag{3.11}$$

Hence, following the previous development, formally, U is a solution of

$$\begin{aligned} -\partial_t U - \sigma \Delta U + H(x, \nabla_x U) - A[\mu, b, f][U] &= \tilde{f}(x, \mu), \\ U(T, x, \mu) &= \tilde{U}_0(x, \mu). \end{aligned} \tag{3.12}$$

Where $\tilde{f}$ and $\tilde{U}_0$ are defined as in the previous section.

Replacing b by what should be the optimal strategies of the players, and reversing time to lighten notations, one obtains the master equation

$$\begin{aligned} \partial_t U - \sigma \Delta U + H(x, \nabla_x U) - A[\mu, -D_p H(\nabla_x U), f][U] &= \tilde{f}, \\ U(0, x, \mu) &= \tilde{U}_0, \end{aligned} \tag{3.13}$$

which is posed on $(0, \infty) \times \mathbb{T}^d \times \mathcal{A}$.

3.5. **Mathematical analysis of the master equation.** This section contains a partial mathematical analysis of the master equation just derived. The notion of monotone solutions introduced in [2, 3] is used to prove some properties of the value functions for such MFG.

Even if the precise nature of the operator A is not established here, it possesses the following properties.

(1) If there is no learning (e.g. if f is constant), then $\nu_t^{m,\mu} = (K_t)_{\#}\mu$ for any m and A is defined on smooth functions on $\mathcal{A} = \mathcal{P}(\mathcal{P}(\mathbb{T}^d))$.
(2) When evaluated on the minimum μ^* of a function $V : \mathcal{P}(\mathcal{P}(\mathbb{T}^d)) \to \mathbb{R}$, one should have $A[\mu^*, b, f][V] \geq 0$ for any b, f.

(3) If, for some smooth function ϕ, V is a function of the form

$$V(\mu) = \int_{\mathcal{P}(\mathbb{T}^d)} \phi(m)\mu(dm), \tag{3.14}$$

then, whatever the function f,

$$A[\mu, b, f][V] = \int_{\mathcal{P}(\mathbb{T}^d)} \left\langle \frac{\delta\phi}{\delta m}, \Delta m - \mathrm{div}(bm) \right\rangle \mu(dm). \tag{3.15}$$

.

The first point is mainly a remark and only the third point is not immediate. It is simply a consequence of the fact that because V is linear, $\mathbb{E}_\mu[V(\nu^{m,\mu}_{dt})] = V((K_t)_{\#}\mu)$.

Although the description of the operator A is quite poor at this time, the properties (2) and (3) above are sufficient to define a concept of monotone solutions here.

The main advantage of monotone solutions is that they allow to define solutions of (3.17) without using the operator A directly on U but on a large set of simpler functions instead, namely functions of the form (3.14). To define precisely these simpler functions, and to make the following more understandable, one needs to use a duality between $\mathcal{C}(\mathbb{T}^d)$ and $\mathcal{P}(\mathcal{P}(\mathbb{T}^d))$. As suggested by the computation done in the blind case, we choose the following duality between $\phi \in \mathcal{C}(\mathbb{T}^d)$ and $\mu \in \mathcal{M}(\mathcal{P}(\mathbb{T}^d))$:

$$\langle \phi, \mu \rangle := \int_{\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d} \phi(x)m(dx)\mu(dm). \tag{3.16}$$

The idea of monotone solutions consists in looking at minima of the function $W(t,\mu) := \langle U(t,\cdot,\mu) - \phi, \mu - \nu\rangle$ for $\phi \in \mathcal{C}^2(\mathbb{T}^d)$ and $\nu \in \mathcal{M}(\mathcal{P}(\mathbb{T}^d))$. Formally, W is a solution of

$$\begin{aligned} &\partial_t W - \sigma\langle \Delta U + H(x, \nabla_x U), \mu - \nu\rangle - A[\mu, -D_pH(\nabla_x U), f][W] \\ &\qquad = \langle \tilde{f}, \mu - \nu\rangle - \langle \sigma\Delta(U - \phi) - D_pH(\nabla_x U)\cdot\nabla_x(U-\phi), \mu\rangle \,. \end{aligned} \tag{3.17}$$

Indeed, remark that

$$A[\mu, -D_pH(\nabla_x U), f][W] = \langle A[\mu, -D_pH(\nabla_x U), f][U], \mu - \nu\rangle + A[\mu, -D_pH(\nabla_x U), f][\Psi] \tag{3.18}$$

where $\Psi : \mu' \to \langle U(\mu) - \phi, \mu'\rangle$. The previous formula is just the equivalent of the formula $(fg)' = f'g + g'f$ for the operator A. Now using (3.15) we deduce that $A[\mu, -D_pH(\nabla_x U), f][\Psi] = \langle \sigma\Delta(U-\phi) - D_pH(\nabla_x U)\cdot\nabla_x(U-\phi), \mu\rangle$.

Using both the facts that on minima of W, $A[\mu, -D_pH(\nabla_x U), f][W] \geq 0$, and that Ψ is linear, we arrive at the

Definition 3.1. *We say that a continuous function $U : [0,T]\times\mathbb{T}^d\times\mathcal{A}$, smooth in its second argument, is a value of the MFG with observed payments and unknown repartition of players if :*

- *for any $\mathcal{C}^2$ function $\phi : \mathbb{T}^d \to \mathbb{R}$, for any measure $\nu \in \mathcal{M}(\mathcal{M}(\mathbb{T}^d))$, for any smooth function $\vartheta : [0,\infty) \to \mathbb{R}$ and any point $(t_0, \mu_0) \in (0, t_0) \times \mathcal{A}$ of minimum of $(t,\mu) \to \langle U(t,\cdot,\mu) - \phi, \mu - \nu\rangle - \vartheta(t)$ on $(0,t_0) \times \mathcal{A}$, the following holds*

$$\begin{aligned}\frac{d\vartheta}{dt}(t_0) + \langle -\sigma\Delta U + H(\cdot, \nabla_x U), \mu_0 - \nu\rangle &\geq \langle \tilde{f}(\cdot,\mu_0), \mu_0 - \nu\rangle \\ &- \langle \sigma\Delta(U-\phi) - D_p H(\nabla_x U)\cdot \nabla_x(U-\phi), \mu_0\rangle .\end{aligned} \tag{3.19}$$

- *the initial condition holds*

$$U(0,x,\mu) = \tilde{U}_0(x,\mu). \tag{3.20}$$

Remark 3.2. *The previous definition only involves the information of the payments through the set $\mathcal{A}$, on which the value function is defined. Defining it on a larger set than $\mathcal{A}$ would be meaningless since such belief are not coherent with the model.*

Ideally, following Remark 2.3, one could hope to establish a uniqueness result for value functions in the sense of Definition 3.1. However because of the nature of the set $\mathcal{A}$, we have not been able to prove such a result in a general framework. The nature of the set $\mathcal{A}$ can be described with following.

Proposition 3.2. *The set $\mathcal{A}$ is compact. As soon as f is neither one-to-one or constant, $\mathcal{A}$ is not convex.*

Proof. Consider $(\mu_n)_{n\geq 0}$, a $\mathcal{A}$ valued converging sequence in $\mathcal{P}(\mathcal{P}(\mathbb{T}^d))$ whose limit is $\bar{\mu}$. For all $n \geq 0$, denote $g_n : \mathbb{T}^d \to \mathbb{R}$ such that μ_n almost everywhere, $f(m) = g_n$. From the assumptions on f, we obtain that $(g_n)_{n\geq 0}$ is compact in $\mathcal{C}(\mathbb{T}^d)$. Because for all $n \geq 0$,

$$g_n = \int_{\mathcal{P}(\mathbb{T}^d)} f(m')\mu_n(dm'), \tag{3.21}$$

passing to the limit in the previous equation, we deduce that $(g_n)_{n\geq 0}$ converges uniformly toward

$$g := \int_{\mathcal{P}(\mathbb{T}^d)} f(m')\bar{\mu}(dm'). \tag{3.22}$$

It easily follows that $\bar{\mu}$ almost everywhere, $f(m) = g$. Thus $\mathcal{A}$ is a closed (compact) set.

If f is neither one-to-one or constant, take m_1, m_2 and m_3 such that $f(m_1) = f(m_2) \neq f(m_3)$ and remark that $\frac{1}{2}\delta_{m_1} + \frac{1}{2}\delta_{m_2}$ and δ_{m_3} belongs to $\mathcal{A}$ while none of their (strict) convex combination does. □

Remark 3.3. *Rigorously, $\mathcal{A}$ is not convex even if f is one-to-one. However in this case, it is isomorphic to $\mathcal{P}(\mathbb{T}^d)$ which is convex.*

Even if we were not able to establish a general result of uniqueness, we could prove the following, which is commented immediately afterwards.

Theorem 3.1. *If f and U_0 are monotone, then two value functions U and V of the MFG, in the sense of Definition 3.1, are such that for any $t \geq 0, \mu \in \mathcal{A}, m, m' \in \mathcal{P}(\mathbb{T}^d)$ such that $f(m) = f(m') = \tilde{f}(\mu)$*

$$\int_{\mathbb{T}^d} U(t,x,\mu) - V(t,x,\mu)(m - m')(dx) = 0. \tag{3.23}$$

Moreover, for any $t \geq 0, m \in \mathcal{P}(\mathbb{T}^d)$, $\nabla_x U(t,\cdot,\delta_m) = \nabla_x V(t,\cdot,\delta_m)$.

Remark 3.4. *The last part of the Theorem only states that when the players know the repartition of players, only one profile of strategies is possible for the players, which is the one in the case with full information. The equality can be improved to equality of the value on Dirac masses following [3] under stronger assumptions on the monotonicity of f. The first part of the result is a bit less classical. It allows to characterize a set to which $U - V$ is orthogonal, at any point t, μ. Remark that the larger is $f^{-1}(\{\tilde{f}(\mu)\})$, the more information we have on the difference.*

Proof. Assume that there exists $t \geq 0, \mu, \nu \in \mathcal{A}$ such that

$$\langle U(t,\mu) - V(t,\nu), \mu - \nu\rangle = -c_0 < 0. \tag{3.24}$$

Then, there exists $\delta > 0$ such that for any α, the function W defined by

$$W(t,s,\mu,\nu) = \langle U(t,\mu) - V(s,\nu), \mu - \nu\rangle + \alpha(t-s)^2 + \delta(t+s) \tag{3.25}$$

is not non-negative, with a minimum lower than $-\frac{1}{2}c_0$. Since $\mathcal{A}$ is compact and W continuous, consider a minimum of W on $[0,T]^2 \times \mathcal{A}^2$ denoted by (t_*, s_*, μ_*, ν_*).

Assume first that $t_*, s_* > 0$, from the definition of the value function, the following holds.

$$\begin{aligned}
&-2\delta - \langle \sigma\Delta U + H(\cdot,\nabla_x U), \mu_* - \nu_*\rangle - \langle \sigma\Delta V + H(\cdot,\nabla_x V), \nu_* - \mu_*\rangle \geq \\
&\quad \langle \tilde{f}(\cdot,\mu_*) - \tilde{f}(\cdot,\nu_*), \mu_* - \nu_*\rangle - \langle U - V, \nabla_m(\sigma\Delta m + \operatorname{div}(D_pH(\nabla_x U)m)\mu_*)\rangle \\
&\quad - \langle V - U, \nabla_m(\sigma\Delta m + \operatorname{div}(D_pH(\nabla_x V)m)\nu_*)\rangle.
\end{aligned} \tag{3.26}$$

The previous is only the addition of the information from Definition 3.1 for U and V. From the convexity of the Hamiltonian and the monotonicity of f, we obtain that $-2\delta > 0$ which is a contradiction. If either t_* or s_* are equal to 0, then using the continuity of U and V, we also obtain a contradiction taking α as big as necessary.

Hence we deduce that for every $\mu, \nu \in \mathcal{A}, t \geq 0$,

$$\langle U(t,\mu) - V(t,\nu), \mu - \nu\rangle \geq 0. \tag{3.27}$$

We now explain how this information translates into the required result. Consider $\bar{\mu} \in \mathcal{A}$, $\theta \in (0,1)$ and two measures $m_1, m_2 \in \mathcal{P}(\mathbb{T}^d)$ such that $f(m_1) = f(m_2) =$

$\tilde{f}(\bar{\mu})$. Using (3.27) for $\mu = (1-\theta)\bar{\mu} + \theta\delta_{m_1}$ and $\nu = (1-\theta)\bar{\mu} + \theta\delta_{m_2}$, we obtain that

$$\theta\langle U(t,\mu) - V(t,\nu), \delta_{m_1} - \delta_{m_2}\rangle \geq 0. \tag{3.28}$$

Dividing by θ and letting $\theta \to 0$ yields

$$\int_{\mathbb{T}^d} U(t,x,\bar{\mu}) - V(t,x,\bar{\mu})(m_1 - m_2)(dx) = 0. \tag{3.29}$$

(The equality is obtained by inverting the role of m_1 and m_2.)

Consider now $m \in \mathcal{P}(\mathbb{T}^d)$ and $\bar{\mu} = \delta_m$. Taking $m_1, m_2 \in \mathcal{P}(\mathbb{T}^d)$, and $\theta \in (0,1)$, define $\mu = \delta_{(1-\theta)m+\theta m_1}$ and $\nu = \delta_{(1-\theta)m+\theta m_2}$, and proceeding as immediately above, we obtain

$$\int_{\mathbb{T}^d} U(t,x,\bar{\mu}) - V(t,x,\bar{\mu})(m_1 - m_2)(dx) = 0, \tag{3.30}$$

which proves the second part of the claim.

□

In the case in which f is strictly monotone, we can obtain more precise information on the difference of two value functions. Before we state the two following results, observe that if f is strictly monotone on $\mathcal{P}(\mathbb{T}^d)$, then $\tilde{f}$ is strictly monotone on $\mathcal{A}$.

Proposition 3.3. *Assume that f is strictly monotone and that U and V are two monotone solutions in the sense of Definition 3.1. Fix $T > 0$, then for $t \leq T$*

$$\sup_{\mu\in\mathcal{A}} \|U(t,\mu) - V(t,\mu)\|_\infty \leq Ct, \tag{3.31}$$

where C only depends on H, f, T and

$$M := \sup_{t\in[0,T],\mu,\nu} \|U(t,\cdot,\mu)\|_{C^2} + \|V(t,\cdot,\nu)\|_{C^2}. \tag{3.32}$$

Proof. Let U and V be two such solutions. Define the constant

$$M := \sup_{t\in[0,T],\mu,\nu} \|U(t,\cdot,\mu)\|_{C^2} + \|V(t,\cdot,\nu)\|_{C^2}. \tag{3.33}$$

Remark that

$$K_s := \sup_{t\in[0,s],\mu\in\mathcal{A}} \|U(t,\mu) - V(t,\mu)\|_\infty = -\inf_{t\in[0,s],\mu\in\mathcal{A},\eta} \langle U(t,\mu) - V(t,\mu), \eta\rangle, \tag{3.34}$$

where the infimum in η is taken over bounded measure of mass 1 on $\mathcal{P}(\mathbb{T}^d)$. Fix an arbitrary horizon $T > 0$ and consider $\delta := \frac{K_T}{2T}$. Define for $\eta \in \mathcal{M}(\mathbb{T}^d), \alpha, \epsilon > 0$,

$$W(t,s,\mu,\nu) = \langle U(t,\mu) - V(s,\nu), \mu - \nu + \epsilon\eta\rangle + \alpha(t-s)^2 + \epsilon\delta(t+s). \tag{3.35}$$

Consider a point $(t_\epsilon, s_\epsilon, \mu_\epsilon, \nu_\epsilon)$ of minimum of W on $[0,T]^2 \times \mathcal{A}^2$. Assume that $t_\epsilon, s_\epsilon > 0$. Because both U and V are value functions of the MFG, we obtain that

(3.36)
$$\begin{aligned}
&-2\epsilon\delta - \langle \sigma\Delta(U-V) + H(\cdot,\nabla_x U) - H(\cdot,\nabla_x V), \mu_\epsilon - \nu_\epsilon + \epsilon\eta\rangle \geq \\
&\quad \langle \tilde{f}(\cdot,\mu_\epsilon) - \tilde{f}(\cdot,\nu_\epsilon), \mu_\epsilon - \nu_\epsilon + \epsilon\eta\rangle - \langle U - V, \nabla_m(\sigma\Delta m + \mathrm{div}(D_pH(\nabla_x U)m)\mu_\epsilon)\rangle \\
&\quad - \langle V - U, \nabla_m(\sigma\Delta m + \mathrm{div}(D_pH(\nabla_x V)m)\nu_\epsilon)\rangle.
\end{aligned}$$

The convexity of the Hamilton yields

$$(3.37)\qquad \begin{aligned}
&-2\epsilon\delta - \langle \sigma\Delta(U-V) + H(\cdot,\nabla_x U) - H(\cdot,\nabla_x V), \epsilon\eta\rangle \geq \\
&\quad \langle \tilde{f}(\cdot,\mu_\epsilon) - \tilde{f}(\cdot,\nu_\epsilon), \mu_\epsilon - \nu_\epsilon + \epsilon\eta\rangle.
\end{aligned}$$

Letting $\epsilon \to 0$ implies that any limit point $(\bar{\mu},\bar{\nu})$ of $(\mu_\epsilon,\nu_\epsilon)_{\epsilon>0}$ satisfies

$$(3.38)\qquad \langle \tilde{f}(\bar{\mu}) - \tilde{f}(\bar{\nu}), \bar{\mu} - \bar{\nu}\rangle = 0.$$

Hence from the strict monotonicity of $\tilde{f}$: $\tilde{f}(\bar{\mu}) = \tilde{f}(\bar{\nu})$, which itself implies $\lim_{\epsilon\to 0} f(\mu_\epsilon) - f(\nu_\epsilon) = 0$ (maybe only along a subsequence). Coming back to (3.37), using the monotonicity of $\tilde{f}$ yields, after dividing by ϵ

$$(3.39)\qquad 2\delta \leq M\langle 1, \eta\rangle + o(\epsilon).$$

This implies the required result since i) the case $t_\epsilon = 0$ or $s_\epsilon = 0$ does not raise any difficulty, ii) T is chosen arbitrary. □

This previous result is an additional step in the direction of a suitable general uniqueness result. In the case of a simple dependence of f in m, the following result of uniqueness holds.

Theorem 3.2. *Assume that f is strictly monotone and $f(m)$ only depends on $m \in \mathcal{P}(\mathbb{T}^d)$ through its first moment $E(m) := \int_{\mathbb{T}^d} xm(dx)$. Then there is at most one value function of the MFG in the sense of Definition 3.1.*

Proof. Consider U and V two value functions of the MFG in the sense of Definition 3.1. Thanks to Theorem 3.1, there exists two functions a and b on : $[0,T] \times \mathcal{A}$ such that $U(t,x,\mu) = V(t,x,\mu) + a(t,\mu)\cdot x + b(t,\mu)$. Fix a time horizon t^* and define

$$(3.40)\qquad \delta = (2t^*)^{-1} \sup_{t\leq t^*,\eta} \langle a(t,\mu)\cdot x, \eta\rangle$$

where the supremum is taken over all $\eta \in \mathcal{M}(\mathcal{M}(\mathbb{T}^d))$ such that $\langle 1,\eta\rangle = 0$ and $\langle 1, |\eta|\rangle \leq 1$. Consider any such $\eta \in \mathcal{M}(\mathcal{M}(\mathbb{T}^d))$ and define

$$(3.41)\qquad W(t,s,\mu,\nu) = \langle U(t,\mu) - V(s,\nu), \mu - \nu + \epsilon\eta\rangle + \alpha(t-s)^2 + \epsilon\delta(t+s).$$

Denote $(t_\epsilon, s_\epsilon, \mu_\epsilon, \nu_\epsilon)$ a point of minimum of W and assume $t_\epsilon, s_\epsilon > 0$. Reasoning as in the proof of Proposition 3.3, we obtain that

$$(3.42)\qquad 2\delta \leq \langle -\sigma\Delta(U-V) + H(\cdot,\nabla_x U) - H(\cdot,\nabla_x V), \eta\rangle + o(\epsilon).$$

By construction of a and b, as well as by the regularity of H, the following holds.

$$2\delta \leq \langle C\langle |a(t_\epsilon,\mu_\epsilon)|,\eta\rangle + o(\epsilon). \tag{3.43}$$

In the previous C is simply the Lipschitz constant of H. Using the properties of η yields

$$2\delta \leq C|a(t_\epsilon,\mu_\epsilon)|. \tag{3.44}$$

The definition of δ leads to

$$\sup_\eta \langle a(t,\mu)\cdot x,\eta\rangle \leq t^* C|a(t_\epsilon,\mu_\epsilon)|. \tag{3.45}$$

and thus

$$\delta \leq Ct^*\delta. \tag{3.46}$$

Hence, if t^* is small enough, compared to C which depends only on H, $\delta = 0$. Since t^* is chosen independently of U and V, it follows that the same argument can be repeated on $[t^*, 2t^*]$, so that $\delta = 0$ for any t^*. This proves that $a = 0$ uniformly. With a similar argument for $\eta = \delta_\lambda$, for λ the Lebesgue on $\mathbb{T}^d$, it follows that $b = 0$ also, and thus that $U = V$, which proves the claim.

□

Remark 3.5. *In the end, the key ingredient which makes the previous result true and forbid us to extend it to general f, is that a control of the type*

$$|\langle -\sigma\Delta(U-V) + H(\cdot,\nabla_x U) - H(\cdot,\nabla_x V),\eta\rangle| \leq C\sup_{\eta'}\langle U-V,\eta'\rangle \tag{3.47}$$

holds for well chosen η' in the supremum above. Hence, the previous result can be extended to situations in which such an estimate on the inverse problem of an Hamitlon-Jacobi-Bellman equation holds, at least for functions U and V which satisfy the conclusion of Theorem 3.1.

3.6. **Comments and future perspectives.** As already mentioned above, this study on MFG with unknown distribution of players and observed payments is not a complete one and, hopefully, more results are to come. The approach proposed here used the notion of monotone solutions of MFG master equations to obtain a definition of solutions (Definition 3.1). If the lack of a general uniqueness result pleads against this notion of solution, this definition is nonetheless helpful to prove several properties of such value functions and a uniqueness result for a particular case. It is possible that a more restrictive notion of solution will prove to be better adapted to this problem.

In the study of this problem, a fundamental question which remains open is the question of the existence of such a value function. Because the effect of the observation of the payments possesses some similarity with the presence of a common noise in MFG, some approaches to prove existence are suggested from the literature on MFG with common noise, maybe the most natural would be the one of [7]. As already mentioned above, a direct application of this approach does not semm feasible. However, if we restrict ourselves to beliefs which are combinations of Dirac masses, such a strategy looks viable. It will then suffice to have a uniform estimate on the continuity of U with respect to μ to pass to the limit. This approach is not presented here because we were not able to establish such a continuity estimate.

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